Frank Merle and Hatem Zaag

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Isolatedness of characteristic points at blow-up for a semilinear wave equation in one space dimension

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We consider the one dimensional semilinear wave equation

\[
\begin{aligned}
\begin{cases}
\partial_t^2 u - \partial_x^2 u + |u|^{p-1}u, \quad &\text{in } \mathbb{R}^+ \\
u(0) = u_0 \text{ and } u_t(0) = u_1, &\text{on } \mathbb{R}
\end{cases}
\end{aligned}
\]

(1)

where \(u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}, p > 1, u_0 \in H^{1}_{\text{loc},u} \) and \(u_1 \in L^{2}_{\text{loc},u} \) with \(\|v\|^{2}_{L^{2}_{\text{loc},u}} = \sup_{a \in \mathbb{R}} \int_{|x-a|<1} |v(x)|^2 \, dx \) and \(\|v\|^{2}_{H^{1}_{\text{loc},u}} = \|v\|^{2}_{L^{2}_{\text{loc},u}} + \|\nabla v\|^{2}_{L^{2}_{\text{loc},u}} \).

The Cauchy problem for equation (1) in the space \(H^{1}_{\text{loc},u} \times L^{2}_{\text{loc},u} \) follows from the finite speed of propagation and the wellposedness in \(H^1 \times L^2 \) (see Ginibre, Soffer and Velo [6]). If the solution is not global in time, then we call it a blow-up solution. The existence of blow-up solutions is guaranteed by ODE techniques together with the finite speed of propagation, or also by the following blow-up criterion from Levine [9]:

If \((u_0, u_1) \in H^1 \times L^2(\mathbb{R})\) satisfies

\[
\int_{\mathbb{R}} \left( \frac{1}{2} |u_1(x)|^2 + \frac{1}{2} |\partial_x u_0(x)|^2 - \frac{1}{p+1} |u_0(x)|^{p+1} \right) \, dx < 0,
\]

then the solution of (1) cannot be global in time.

More blow-up results can be found in Caffarelli and Friedman [5], [4], Alinhac [1] and Kichenassamy and Littman [7], [8].

If \(u\) is a blow-up solution of (1), we define (see for example Alinhac [1]) a 1-Lipschitz curve \(\Gamma = \{(x, T(x))\}\) such that \(u\) cannot be extended beyond the set called the maximal influence domain of \(u\):

\[
D = \{(x,t) \mid t < T(x)\}.
\]

(2)

\(\bar{T} = \inf_{x \in \mathbb{R}} T(x)\) and \(\Gamma\) are called the blow-up time and the blow-up graph of \(u\).

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A point $a$ is a non characteristic point (or a regular point) if there are $\delta_0 \in (0,1)$ and $t_0 < T(a)$ such that $u$ is defined on $C_{a,T(a),\delta_0} \cap \{t \geq t_0\}$ where

$$C_{\bar{x},\bar{t},\bar{\delta}} = \{(x,t) \mid t < \bar{t} - \bar{\delta}|x - \bar{x}|\}.$$

If not, then we call $a$ a characteristic point (or a singular point). Naturally, we denote by $R$ (resp. $S$) the set of non characteristic (resp. characteristic) points. Note then that $R \cup S = \mathbb{R}$.

In our papers [14], [16], [15], [17], [18], [19] and [20], we made several contributions to the study of blow-up solutions of (1), namely the description of its blow-up graph and blow-up behavior in selfsimilar variables.

1 The blow-up graph of equation (1)

It is clear from rather simple arguments that $R \neq \emptyset$ for any blow-up solution $u(x,t)$ (if $T(x)$ achieves its minimum at $a$, then $a \in R$; if the infimum of $T(x)$ is at infinity, then there exists a large $a$ such that $a \in R$ as we state in the remark following Theorem 1 in [18]). On the contrary, the situation was unclear for $S$, and it was commonly conjectured
before our contributions that $S$ was empty. In particular, that was the case in the examples constructed by Caffarelli and Friedman in [5] and [4]. In [19], we prove that the conjecture was false. More precisely, we proved the following (see Proposition 1 in [19]):

**Proposition 1 (Existence of initial data with $S \neq \emptyset$)** If the initial data $(u_0, u_1)$ is odd and $u(x, t)$ blows up in finite time, then $0 \in S$.

![Fig 3: Odd initial data which makes the origin a characteristic point](image)

As we explicitly show in Theorem 4 below, the existence of characteristic points is linked to sign changes in the solution near the singular point $(a, T(a))$. This enables us to give the following criterion for the non existence of characteristic points on some finite interval (see Theorem 4 in [19]):

**Proposition 2 (Non existence of characteristic points)** Consider $u(x, t)$ a blow-up solution of (1) such that $u(x, t) \geq 0$ for all $x \in (a_0, b_0)$ and $t_0 \leq t < T(x)$ for some real $a_0, b_0$ and $t_0 \geq 0$. Then, $(a_0, b_0) \subset R$.

**Remark**: This result can be seen as a generalization of the result of Caffarelli and Friedman [5] and [4], with no restriction on initial data. Indeed, from our result, taking non-negative initial data suffices to exclude the occurrence of characteristic points.

For general blow-up solutions, we proved the following facts about $R$ and $S$ in [18] and [19] (see Theorem 1 (and the following remark) in [18], see Theorems 1 and 2 in [20]):

**Theorem 3 (Geometry of the blow-up graph)**

(i) $R$ is a non empty open set, and $x \mapsto T(x)$ is of class $C^1$ on $R$;

(ii) $S$ is made of isolated points, and given $a \in S$, if $0 < |x - a| \leq \delta_0$, then

$$\frac{1}{C_0 |\log(x - a)|^{(k(a) - 1)(p - 1) / 2}} \leq T'(x) + \frac{x - a}{|x - a|} \leq \frac{C_0}{|\log(x - a)|^{(k(a) - 1)(p - 1) / 2}}$$

for some $\delta_0 > 0$ and $C_0 > 0$, where $k(a) \geq 2$ is an integer. In particular, $T(x)$ is right and left differentiable at $a$, with $T'_l(a) = 1$ and $T'_r(a) = -1$.  

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Remark: In [21], Nouaili improves the regularity of the restriction of \( x \mapsto T(x) \) to \( \mathbb{R} \) to \( C^{1,\alpha} \) for some \( \alpha > 0 \).

Remark: Integrating the estimate of (ii) in Theorem 3, we see that the blow-up set is corner-shaped near \( a \) in the sense that

\[
\frac{|x - a|}{C_0 |\log(x - a)|^{(k(a)-1)(p-1)}} \leq T(x) - T(a) + |x - a| \leq C_0 \frac{|x - a|}{|\log(x - a)|^{(k(a)-1)(p-1)}}. \tag{3}
\]

Fig. 4: The blow-up set is corner-shaped near characteristic points

In particular, there exists no solution of the semilinear wave equation (1) with a characteristic point \( a \) such that \( T(x) \) is differentiable at \( x = a \).

Note from (3) that the blow-up set never touches the backward light cone with vertex \((a, T(a))\) (except of course at \( a \)), and that the distance between them is bounded from above and from below by the same rate, which is quantified in terms of the integer \( k(a) \geq 2 \).

In particular, from the shape of the solution near \((a, T(a))\), we can recover the integer \( k(a) \geq 2 \), and \( k(a) - 1 \) is the number of sign changes of the solution near \((a, T(a))\) as we will see in (ii) of Theorem 4 below. In one word, the shape of the solution near \((a, T(a))\) gives the topology of the solution and conversely.

Remark: The fact that the elements of \( S \) are isolated points is not elementary. Direct arguments give no more than the fact that \( S \neq \mathbb{R} \) (a point \( a \) such that \( T(a) \) is the blow-up time is non characteristic). The first step of the proof is done in [19] where we proved that \( S \) has an empty interior and that in similarity variables, the solution splits in a non trivial decoupled sum of (at least 2) solitons with alternate signs (see (ii) of Theorem 4 below for a statement). The second step is done in [20]. It consists in using this decomposition and a good understanding of the dynamics of the equation in similarity variables (see equation (5) below) near a decoupled sum of “generalized” solitons. In fact, this is the first time where flows near an unstable sum of solitons are used and where such a result is obtained.

Remark: The fact that \( S \) is made of isolated points certainly does not hold in general for quasilinear wave equations. Indeed, in [2], Alinhac gives an explicit solution \( u(x, t) \) for the following nonlinear wave equation

\[
\partial_t^2 u = \partial_x^2 u + \partial_x u \partial_t u,
\]
whose domain of definition is
\[ D = \mathbb{R} \times [0, \infty) \setminus \{(x, t) \mid t \geq 1, |x| \leq t - 1\} \]
(when \( 0 \leq t < 1 \), \( u(x, t) = 4 \arctan \left( \frac{x}{t-1} \right) \)). In this example, we clearly see that \( \mathcal{R} = \{0\} \), \( \mathcal{S} = \mathbb{R}^* \), and the boundary of \( D \) is characteristic (i.e. has slope \( \pm 1 \)) on \( \mathcal{S} \).

\section{Asymptotic behavior near the blow-up graph}

As one may guess from the above description, the asymptotic behavior will not be the same on \( \mathcal{R} \) and on \( \mathcal{S} \). In both cases, we need to use the similarity variables which we recall in the following. Let us stress the fact that the keystone of our work is the existence of a Lyapunov functional in similarity variables.

Given some \( a \in \mathbb{R} \), we introduce the following self-similar change of variables:
\[
 w_a(y, s) = (T(a) - t)^{-p+1} u(x, t), \quad y = \frac{x - a}{T(a) - t}, \quad s = -\log(T(a) - t). \tag{4}
\]

The function \( w = w_a \) satisfies the following equation for all \( y \in B = B(0,1) \) and \( s \geq -\log T(a) \):
\[
 \partial^2_{ss} w = \mathcal{L} w - 2(p+1) \frac{w}{(p-1)^2} + |w|^{p-1} w - \frac{p+3}{p-1} \partial_s w - 2 y \partial^2_{y,y} w \tag{5}
\]
where \( \mathcal{L} w = \frac{1}{\rho} \partial_y \left( \rho(1-y^2) \partial_y w \right) \) and \( \rho(y) = (1 - y^2)^{\frac{2}{p-1}} \).

From Antonini and Merle \cite{3}, we know the existence of the following Lyapunov functional for equation (5):
\[
 E(w) = \int_{-1}^{1} \left( \frac{1}{2} \partial_s w^2 + \frac{1}{2} \partial_y w^2 \right) \left( 1 - y^2 \right) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \rho dy, \tag{7}
\]
defined for \( \partial_s w, w \in \mathcal{H} \) where
\[
 \mathcal{H} = \left\{ q \mid \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left( q_1^2 + (q_1')^2 \left( 1 - y^2 \right) + q_2^2 \right) \rho dy < +\infty \right\}. \tag{8}
\]

Using this energy structure together with interpolation and the Gagliardo-Nirenberg estimate, we proved in \cite{14}, \cite{16}, \cite{15} and \cite{17} that \( \{w_a(s), \partial_s w_a(s)\} \) is bounded in the energy space \( \mathcal{H} \). Moreover, if \( a \in \mathcal{R} \), then the bound holds in \( H^1 \times L^2(-1,1) \) as well by a covering technique.

From Proposition 1 in \cite{17}, we know that the only stationary solutions of (5) in the space \( \mathcal{H} \) are \( q \equiv 0 \) or \( w(y) \equiv \pm \kappa(d, y) \), where \( d \in (-1,1) \) and
\[
 \kappa(d, y) = \kappa_0 \left( 1 - d^2 \right)^{\frac{1}{p-1}} \left( 1 + dy \right)^{-\frac{2}{p-1}} \quad \text{where} \quad \kappa_0 = \left( \frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}} \quad \text{and} \quad |y| < 1. \tag{9}
\]

We will sometimes refer to \( \pm \kappa(d, y) \) as “solitons”. Note that the set of stationary solutions is made of 3 connected components. One wanders whether these stationary solutions are
good candidates for the convergence of \( w_a(y,s) \), at least when \( a \in \mathcal{R} \). In the context of Hamiltonian systems, this question is delicate, and there is no natural reason for such a convergence, since the wave equation is time reversible. This kind of difficulties is also encountered for the critical generalized Korteweg de Vries equation (see Martel and Merle [10]) and for the Nonlinear Schrödinger equation (see Merle and Raphaël [11]).

As a matter of fact, there is convergence for \( w_a \) when \( a \in \mathcal{R} \) as we see from the following result (see Corollary 4 in [17] and Theorem 6 in [19]):

**Theorem 4 (Asymptotic behavior near the blow-up graph)**

(i) **Case where** \( a \in \mathcal{R} \): **Existence of an asymptotic profile.** There exist \( \delta_0(a) > 0 \), \( |e(a)| = 1 \), \( s_0(a) \geq -\log T(a) \) such that for all \( s \geq s_0 \):

\[
\left\| \left( \frac{w_a(s)}{\partial_s w_a(s)} \right) - e(a) \begin{pmatrix} \kappa(T'(a),\cdot) \\ 0 \end{pmatrix} \right\|_{H^1} \leq C_0 e^{-\mu_0(s-s_0)}
\]  

for some positive \( \mu_0 \) and \( C_0 \) independent from \( a \). Moreover, \( E(w_a(s)) \rightarrow E(\kappa_0) \) as \( s \rightarrow \infty \).

(ii) **Case where** \( a \in S \): **Decomposition into a sum of decoupled solitons.** It holds that

\[
\left\| \left( \frac{w_a(s)}{\partial_s w_a(s)} \right) - \sum_{i=1}^{k(a)} e_i^*(a) \kappa(d_i(s),\cdot) \right\|_{H^1} \rightarrow 0 \text{ and } E(w_a(s)) \rightarrow k(a)E(\kappa_0)
\]  

as \( s \rightarrow \infty \), for some

\( k(a) \geq 2, \ e_i^*(a) = e_i^*(a)(-1)^{i+1} \) \hfill (12)

and continuous \( d_i(s) = -\tanh \zeta_i(s) \in (-1,1) \) for \( i = 1,\ldots,k(a) \). Moreover, for some \( C_0 > 0 \), for all \( i = 1,\ldots,k(a) \) and \( s \) large enough,

\[
\left| \zeta_i(s) - \left( i - \frac{k(a) + 1}{2} \right) \frac{(p-1)}{2} \log s \right| \leq C_0.
\]  

**Remark:** It happens that (i) holds uniformly for all \( b \) in some neighborhood of \( a \). Therefore, using the Sobolev injection in one dimension and a covering technique, we can show that the convergence of (10) holds in \( L^\infty(-1,1) \) in the sense that

\[
\|w_a(y,s) - e(a)\kappa(T'(a),y)\|_{L^\infty(-1,1)} \rightarrow 0 \text{ as } x \rightarrow \infty.
\]

**Remark:** To better see that the solitons in (11) are decoupled, we should use the change of variables

\[ y = \tanh \xi \text{ where } \xi \in \mathbb{R} \]

and introduce

\( \bar{w}_a(\xi,s) = (1-y^2)^{1/2} w_a(y,s) \) with \( y = \tanh \xi \) and \( \zeta_i(s) = -\tanh^{-1} d_i(s) \).

In this case, estimate (11) yields the fact that

\[
\|\bar{w}_a(\xi,s) - e_i^*(a) \sum_{i=1}^{k(a)} (-1)^i \cosh -\frac{2}{p-1}(\xi - \zeta_i(s))\|_{H^1 \cap L^\infty(\mathbb{R})} \rightarrow 0 \text{ as } s \rightarrow \infty,
\]
It is worth noting that the $\zeta_i(s)$ satisfy a Toda lattice system:

$$\frac{1}{c_1} \zeta_i'(s) = e^{-\frac{2}{p-1}(\zeta_i(s)-\zeta_{i-1}(s))} - e^{-\frac{2}{p-1}(\zeta_{i+1}(s)-\zeta_i(s))} + R_i(s)$$

with

$$R_i(s) = o\left(\sum_{j=1}^{k-1} e^{-\frac{2}{p-1}(\zeta_j(s)-\zeta_i(s))}\right)$$
as $s \to \infty$.

An important step of the proof is to derive this system which gives two facts: the signs $e_i^*$ in (12) are alternate and the solitons’ centers $\zeta_i(s)$ are equidistant (up to a constant) as written in (13).

**Remark:** Note that some elements in the description in similarity variables given above have a geometrical interpretation:
- when $a \in \mathcal{R}$, the solution in similarity variables converges to the profile $e(a)\kappa(T'(a),\cdot)$ which has the slope $T'(a)$ as a parameter;
- when $a \in \mathcal{S}$, $k(a) \geq 2$ is the number of solitons in the decomposition (11) appears also in the upper bound estimate on $T(x)$ for $x$ near $a$ given in (3).

**Remark:** The proof of the convergence in (i) has two major difficulties:
- the linearized operator of equation (5) around the profile $\kappa(d, y)$ is not selfadjoint, which makes the standard tools inefficient for the control of the negative part of the spectrum. Fortunately, the Lyapunov functional structure will be useful in this matter;
- all the non zero stationary solutions of equation (5) are non isolated, which generates a null eigenvalue difficult to control in the linearization of equation (5) around $\kappa(d, y)$. A modulation technique is then used to overcome this difficulty.

Extending the definition of $k(a)$ defined for $a \in \mathcal{S}$ after (11) by setting

$$k(a) = 1 \text{ for all } a \in \mathcal{R},$$

we proved the following energy criterion in [19] and using the monotonicity of the Lyapunov functional $E(w)$, we have the following consequence from the blow-up behavior in Theorem 4:

**Proposition 2.1 (An energy criterion for non characteristic points; see Corollary 7 in [19])**

(i) For all $a \in \mathbb{R}$ and $s_0 \geq -\log T(a)$, we have

$$E(w_a(s_0)) \geq k(a)E(\kappa_0).$$

(ii) If for some $a$ and $s_0 \geq -\log T(a)$, we have

$$E(w_a(s_0)) < 2E(\kappa_0),$$

then $a \in \mathcal{R}$. 

\[\text{Exp. n° XI— Isolatedness of characteristic points at blow-up for a semilinear wave equation}\]
3 A Liouville theorem and a trapping result near the set of stationary solutions

The following Liouville Theorem is crucial in our analysis:

**Theorem 5 (A Liouville Theorem for equation (5))** Consider \( w(y, s) \) a solution to equation (5) defined for all \( (y, s) \in (-\frac{1}{\delta}, \frac{1}{\delta}) \times \mathbb{R} \) such that for all \( s \in \mathbb{R} \),

\[
\|w(s)\|_{H^1(-\frac{1}{\delta}, \frac{1}{\delta})} + \|\partial_s w(s)\|_{L^2(-\frac{1}{\delta}, \frac{1}{\delta})} \leq C^* \tag{14}
\]

for some \( \delta > 0 \) and \( C^* > 0 \). Then, either \( w \equiv 0 \) or \( w \) can be extended to a function (still denoted by \( w \)) defined in

\[
\{ (y, s) \mid -1 - T_0 e^s < d_0 y \} \supset \left( -\frac{1}{\delta_\ast}, \frac{1}{\delta_\ast} \right) \times \mathbb{R} \text{ by } w(y, s) = \theta_0 \kappa_0 \frac{(1 - d_0^2)^{1-p}}{(1 + T_0 e^s + d_0 y)^{p-1}}, \tag{15}
\]

for some \( T_0 \geq T^\ast, d_0 \in [-\delta_\ast, \delta_\ast] \) and \( \theta_0 = \pm 1 \), where \( \kappa_0 \) defined in (9).

**Remark:** Note that deriving blow-up estimates through the proof of Liouville Theorems has been successful for different problems. For the case of the heat equation

\[
\partial_t u = \Delta u + |u|^{p-1} u \tag{16}
\]

where \( u : (x, t) \in \mathbb{R}^N \times [0, T) \to \mathbb{R}, p > 1 \) and \((N-2)p < N + 2\), the blow-up time \( T \) is unique for equation (16). The blow-up set is the subset of \( \mathbb{R}^N \) such that \( u(x, t) \) does not remain bounded as \((x, t)\) approaches \((a, T)\). In [22], the second author proved the \( C^2 \) regularity of the blow-up set under a non degeneracy condition. A Liouville Theorem proved in [13] and [12] was crucially needed for the proof of the regularity result in the heat equation. The above Liouville Theorem is crucial for the regularity of the blow-up set for the wave equation (Theorem 4).

The second fundamental crucial result for our contributions is given by the following trapping result from [17] (See Theorem 3 in [17] and its proof):

**Proposition 3.1 (A trapping result near the sheet \( d \mapsto \kappa(d, y) \) of stationary solutions)** There exists \( \epsilon_0 > 0 \) such that if \( w \in C([s^*, \infty), \mathcal{H}) \) for some \( s^* \in \mathbb{R} \) is a solution of equation (5) such that

\[
\forall s \geq s^*, \quad E(w(s)) \geq E(\kappa_0) \text{ and } \left\| \begin{pmatrix} w(s^*) \\ \partial_s w(s^*) \end{pmatrix} \right\|_{\mathcal{H}} \leq \epsilon^* \tag{17}
\]

for some \( d^* = -\tanh \xi^*, \omega^* = \pm 1 \) and \( \epsilon^* \in (0, \epsilon_0] \), then there exists \( d_\infty = -\tanh \xi_\infty \) such that

\[
|\xi_\infty - \xi^*| \leq C_0 \epsilon^* \text{ and } \left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} \right\|_{\mathcal{H}} \to 0. \tag{18}
\]
References


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