Centre de Mathématiques Laurent Schwartz
ECOLE POLYTECHNIQUE

SEMINAIRE

Equations aux Dérivées Partielles

2006-2007

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Séminaire É. D. P. (2006-2007), Exposé n° XXI, 19 p.

<http://sedp.cedram.org/item?id=SEDP_2006-2007____A21_0>

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Local Smoothness of Weak Solutions to the Magnetohydrodynamics Equations via Blowup Methods

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Abstract

We demonstrate that there exist no self-similar solutions of the incompressible magnetohydrodynamics (MHD) equations in the space $L^3(\mathbf{R}^3)$. This is a consequence of proving the local smoothness of weak solutions via blowup methods for weak solutions which are locally L^3. We present the extension of the Escauriaza-Seregin-Sverak method to MHD systems.

1 Introduction and Main Results

In this paper we are concerned with the regularity of solutions or existence of the finite time blow up for the principal system of magneto hydro dynamics (MHD):

$$\left.\begin{array}{c} \partial_t v + (v\cdot\nabla)v - \Delta v + \nabla p = \operatorname{rot} H \times H \\ \operatorname{div} v = 0 \end{array}\right\} \quad \text{in } Q_T, \tag{1.1}$$

$$\left.\begin{array}{c} \partial_t H - \Delta H = \operatorname{rot}(v \times H) \\ \operatorname{div} H = 0 \end{array}\right\} \quad \text{in } Q_T. \tag{1.2}$$

Here Ω is a domain in $\mathbb{R}^3$, $Q_T = \Omega \times (-T, 0)$, unknowns are the velocity field $v : Q_T \to \mathbb{R}^3$, pressure $p : Q_T \to \mathbb{R}$, and the magnetic field $H : Q_T \to \mathbb{R}^3$. We use the notation $(v\cdot\nabla)v = v_{i,j}v_j$ where $v_{i,j} \equiv \frac{\partial v_i}{\partial x_j}$ and summation over repeated indexes from 1 to 3 is assumed.

The system above can be interpreted as the usual Navier-Stokes equations perturbed by an additional external force $\operatorname{rot} H \times H$ which is governed by the linear system (1.2), and, intuitively, can be not very regular. In other words, in such a way we relax the original Navier-Stokes system in order to get more freedom in our search for solutions to the Navier-Stokes system which allow the finite-time blow up.

The MHD system possess the same group of symmetries with respect to the natural scaling as the original Navier-Stokes equations. This means that if we consider three functions (v, H, p) satisfying the MHD system (1.1), (1.2) and construct for $\lambda > 0$ the functions

$$v^\lambda(x,t) = \lambda v(\lambda x, \lambda^2 t), \quad H^\lambda(x,t) = \lambda H(\lambda x, \lambda^2 t), \quad p^\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$$

then the functions $(v^\lambda, H^\lambda, p^\lambda)$ form a solution to the MHD equations again. For the systems with symmetries one of the natural ways to find a singular solution is to construct a self-similar solution (i.e. a solution for which $v^\lambda = v$, $H^\lambda = H$, and $p^\lambda = p$) that blows up in finite time. In the case of the MHD system a self-similar solution blowing up at the moment of time $t = t_*$ is given by the formulas:

$$v(x,t) = \lambda(t) U(\lambda(t)x), \quad p(x,t) = \lambda^2(t) P(\lambda(t)x), \quad H(x,t) = \lambda(t) B(\lambda(t)x),$$

where

$$\lambda(t) = \frac{1}{\sqrt{2(t_* - t)}}.$$

If (v, H, p) satisfy (1.1), (1.2) then the functions (U, P, B) must satisfy the stationary system "of the MHD-type":

$$\left.\begin{array}{c} -\Delta U + (x+U)\cdot\nabla U + U + \nabla P = B \cdot \nabla B \\ -\Delta B + (x+U)\cdot\nabla B + B = B\cdot\nabla U \end{array}\right\} \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

Here we used notations $(x+U)\cdot\nabla U = (U_{i,j}U_j + U_{i,j}x_j)$.

The system (1.3) includes additional terms $(x\cdot\nabla)U$ and $(x\cdot\nabla)B$ that "shift" the linear systems to the "spectral" area and make this system hard to analyze. Nevertheless, the initial information on the functions (v, H, p) (such as, for example, the assumption on the local boundedness of the energy) provides some information about the decay of (U, B, P) at infinity, and so, if the original MHD system had only smooth solutions then the functions (U, B) must vanish identically.

The hypotheses on existence of a self-similar blow up solutions for the Navier-Stokes system (i.e. in the case of $H \equiv 0$) was proposed by J. Leray [6] in 1934. This hypotheses was controverted in 1996 by Necas, Růžička, and Šverak in [8]. In particular, in [8] they have proved that any solution of the corresponding system

$$-\Delta U + (x+U)\cdot\nabla U + U + \nabla P = 0 \tag{1.4}$$

must vanish identically providing $U \in L_3(\mathbb{R}^3)$. Later, in [13] Tsai generilized their result by proving that the same assertion is true providing $U \in L_q(\mathbb{R}^3)$, for any $q < +\infty$. The approach of [8], [13] is based on the fact that if U satisfies (1.4) then the auxiliary Bernulli-type function

$$\Pi(x) = \frac{1}{2}|U|^2 + P + (x \cdot U)$$

must satisfy the differential inequality

$$-\Delta\Pi + (x + U) \cdot \nabla\Pi \leq 0$$

and hence the following maximum principle holds:

$$\max_{|x|\leq R} \Pi(x) \leq \max_{|x|=R} \Pi(x) \to 0, \qquad \text{as} \quad R \to \infty,$$

providing one can prove appropriate decay of U at infinity. This maximum principle happens to be crucial steep for both approaches [8] and [13].

Unfortunately, for the system (1.2) we were not able to find an appropriate function Π to provide the maximum principle for it. So, in the case of the MHD equations the method of NRS seems to be not applicable.

An alternative proof of the result of [8] follows from the results of Escuriaza, Seregin, and Šverak [2] on the regularity of the $L_{3,\infty}-$ solutions to the Navier-Stokes system. Namely, denote by $L_{s,r}(Q_T)$ the anisotropic Lebegue space $L_{s,r}(Q_T) \equiv L_r(-T, 0; L_s(\Omega))$. It is clear that the assumption $U \in L_3(\mathbb{R}^3)$ provides the inclusion $v \in L_\infty(-T, 0; L_3(\mathbb{R}^3))$. But according to [2] such solutions are necessary smooth and hence U must vanish identically. The advantage of this approach is that it avoids usage of the maximum principle involved in [8] and [13].

In the present paper we adopt the method established in [2] for the MHD system. So, our result can be interpreted in two ways: first we generalize the $L_{3,\infty}-$ theory for the case of the MHD and second we prove result on the absence of the self-similar blow up for the MHD system which is analogues to the result of [8].

Denote by $Q(R)$ the parabolic cylinder $Q(R) := B(R) \times (-R^2, 0)$ and $Q \equiv Q(1)$. Denote also $W^{1,0}_q(Q) := \{w \in L_q(Q) : \nabla w \in L_q(Q)\}$. The main results of the present paper are the following two theorems:

Theorem 1.1 *Assume v, $H \in L_{2,\infty}(Q) \cap W^{1,0}_2(Q)$, $p \in L_{\frac{3}{2}}(Q)$ satisfy (1.1), (1.2) in Q in the sense of distributions. Assume additionally*

$$v, \ H \in L_{3,\infty}(Q).$$

Then v, H are Hölder continuous on $\bar{Q}(\frac{1}{2})$.

Theorem 1.2 *There are no nontrivial solutions to (1.3) with U, $B \in L_3(\mathbb{R}^3)$.*

It can be interesting to compare our results with the following theorem of Ladyzhenskaya-Prodi-Serrin- type (LPS- condition). The theorem below is the local version of results obtained earlier in [14].

Theorem 1.3 *Assume* v*,* $H \in L_{2,\infty}(Q) \cap W^{1,0}_2(Q)$*,* $p \in L_{\frac{3}{2}}(Q)$ *satisfy (1.1), (1.2) in* Q*. Assume additionally that* v *satisfies the Ladyzhenskaya-Prodi-Serrin condition:*

$$v \in L_{s,r}(Q), \qquad \frac{3}{s} + \frac{2}{r} = 1, \quad s > 3. \tag{1.5}$$

Then v*,* H *are Hölder continuous on* $\bar{Q}(\frac{1}{2})$*.*

As we see, that there is a gap in the regularity theory between the critical and the non-critical LPS- conditions for v. Namely, in the case of $v \in L_{s,r}(Q)$ with $\frac{3}{s} + \frac{2}{r} = 1$, $s > 3$ we do not need any additional information on the magnetic field H besides H belongs to the energy class $L_{2,\infty}(Q) \cap W^{1,0}_2(Q)$. In contrast, in the "critical" case $v \in L_{3,\infty}(Q)$ we require the additional inclusion $H \in L_{3,\infty}(Q)$ to be satisfied. This phenomena is connected only with the linear equation (1.2) and it is due to the lack of the absolute continuity of the $L_{3,\infty}$-norm.

Our paper is organized as follows. In Section 2 we prove Theorem 1.3. In Section 3 we give the notion of suitable weak solutions for the MHD equations and recall some known facts concerning partial regularity for such solutions. In Section 4 we prove Theorem 1.1. Our approach essentially follows to the method developed in [2].
In this paper we use the following notations:

- $B(x_0, R) := \{x \in \mathbb{R}^3 : |x - x_0| < R\}$, $B_R \equiv B(R) := B(0, R)$, $B := B(1)$, $Q(z_0, R) := B(x_0, R) \times (t_0 - R^2, t_0)$ for $z_0 = (x_0, t_0)$, $Q := Q(0, 1)$,
- $\Pi := \mathbb{R}^3 \times (-\infty, 0)$, $\Pi_T := \mathbb{R}^3 \times (-T, 0)$, $Q_{R,T} := B(R) \times (-T, 0)$,
- $\forall\ a$, $b \in \mathbb{R}^n$ denote by $a \otimes b$ the $n \times n$- matrix $(a_i b_j)$,
- $J^1_q(\Omega) := \{w \in W^1_q(\Omega; \mathbb{R}^3) : \operatorname{div} w = 0\}$, $\overset{\circ}{J}{}^1_q(\Omega) = J^1_q(\Omega) \cap \overset{\circ}{W}{}^1_q(\Omega; \mathbb{R}^3)$, $J^1_{q,\nu}(\Omega) = \{w \in J^1_q(\Omega) : w \cdot \nu|_{\partial\nu} = 0\}$ (ν is the outer normal to $\partial\Omega$),
- $L_{s,r}(Q_T) = L_r(-T, 0; L_s(\Omega))$, $\|f\|_{L_{s,r}(Q_T)} \equiv \left(\int_{-T}^0 \|f(\cdot, t)\|^r_{L_s(\Omega)}\, dt\right)^{1/r}$,
- $L_{s,\infty}(Q_T) = L_\infty(-T, 0; L_s(\Omega))$, $\|f\|_{L_{s,\infty}(Q_T)} \equiv \operatorname*{esssup}_{t \in (-T,0)} \|f(\cdot, t)\|_{L_s(\Omega)}$,
- $W^{1,0}_q(Q_T) = \{v, \nabla v \in L_q(Q_T)\}$, $\|u\|_{W^{1,0}_q(Q_T)} := \|u\|_{L_q(Q_T)} + \|\nabla u\|_{L_q(Q_T)}$,
- $W^{2,1}_q(Q_T) = \{v \in W^{1,0}_q(Q_T), \nabla^2 v, \partial_t v \in L_q(Q_T)\}$, $\|u\|_{W^{2,1}_q(Q_T)} := \|u\|_{W^{1,0}_q(Q_T)} + \|\nabla^2 u\|_{q,Q_T} + \|\partial_t u\|_{q,Q_T}$.

2 Proof of the LPS- condition

In this section we present the proof of Theorem 1.3

Lemma 2.1 *Assume v satisfies (1.5) and $H \in L_{2,\infty}(Q) \cap W^{1,0}_2(Q)$ meets the identity*

$$\partial_t H - \Delta H = \operatorname{rot}(v \times H).$$

Then

$$H \in L_{3,\infty}(Q(\tfrac{1}{2})), \qquad |H|^{\frac{3}{2}} \in W^{1,0}_2(Q(\tfrac{1}{2})).$$

Proof: Let $\zeta \in C^\infty(\bar{Q})$ be a cut-off function such that $\zeta \equiv 1$ on $Q(\frac{1}{2})$ and ζ vanishes near $\partial' Q$. Denote by $\tilde{H} \equiv \zeta H$. Then

$$\partial_t \tilde{H} - \Delta \tilde{H} = \operatorname{rot}(v \times \tilde{H}) + F, \tag{2.1}$$

where

$$F = H(\partial_t \zeta - \Delta \zeta) - 2(\nabla H)\nabla\zeta + (v \times H) \times \nabla \zeta.$$

By Hölder inequality we obtain

$$\|v \times H\|^2_{L_2(B)} \le C\|v\|^2_{L_s(B)} \|H\|^2_{L_{\frac{2s}{s-2}}(B)} \le C\left(\|v\|^r_{L_s(B)} + \|H\|^{\frac{2r}{r-2}}_{L_{\frac{2s}{s-2}}(B)} \right).$$

By interpolation $L_{2,\infty}(Q) \cap W^{1,0}_2(Q) \hookrightarrow L_{s_1,r_1}(Q)$ for any $\frac{3}{s_1} + \frac{2}{r_1} = \frac{3}{2}$ we obtain $H \in L_{s_1,r_1}(Q)$ for any s_1, r_1 such that $\frac{3}{s_1} + \frac{2}{r_1} = \frac{3}{2}$. In particular, we can take $s_1 = \frac{2s}{s-2}$, $r_1 = \frac{2r}{r-2}$ as

$$3\ \tfrac{s-2}{2s} + 2\ \tfrac{r-2}{2r} = \tfrac{3}{2} - \tfrac{3}{s} + 1 - \tfrac{2}{r} = \tfrac{5}{2} - \left(\tfrac{3}{s} + \tfrac{2}{r}\right) = \tfrac{3}{2}.$$

So, we conclude that $v \times H \in L_2(Q)$ and

$$\|F\|_{L_2(Q)} \le C\left(\|H\|_{L_2(Q)} + \|v \times H\|_{L_2(Q)}\right).$$

Multiplying (2.1) by $|\tilde{H}|\tilde{H}$ and integrating over $\mathbb{R}^3$ we obtain

$$\frac{1}{3}\frac{d}{dt}\|\tilde{H}\|^3_{L_3(B)} + \int\limits_B |\tilde{H}||\nabla\tilde{H}|^2\, dx \le C\int\limits_B |v||\tilde{H}||\nabla\tilde{H}|\, dx + C\int\limits_B |F||\tilde{H}|^2\, dx. \tag{2.2}$$

By Hölder inequality we estimate

$$\int\limits_B |v||\tilde{H}||\nabla\tilde{H}|\, dx \le \frac{1}{4}\int\limits_B |\tilde{H}||\nabla\tilde{H}|^2\, dx + C\int\limits_B |v|^2|\tilde{H}|^3\, dx$$

The second term in the RHS of the last inequality we again estimate by Hölder inequality:

$$\int\limits_B |v|^2|\tilde H|^3\ dx \le \|v\|^2_{L_s(B)}\|\tilde H\|^3_{L_{\frac{3s}{s-2}}(B)}.$$

Now define $\lambda = \frac{3}{s}$. Then $\lambda \in (0,1)$ and $\frac{s-2}{3s} = \frac{1-\lambda}{3} + \frac{\lambda}{9}$. Using the interpolation inequality $\|\tilde H\|_{L_{\frac{3s}{s-2}}(B)} \le \|\tilde H\|^{1-\lambda}_{L_3(B)}\|\tilde H\|^{\lambda}_{L_9(B)}$ with the help of the imbedding theorem $\|\tilde H\|^{\frac32}_{L_9(B)} = \||\tilde H|^{\frac32}\|_{L_6(B)} \le C\|\nabla|\tilde H|^{\frac32}\|_{L_2(B)}$ we arrive at the estimate

$$\int\limits_B |v|^2|\tilde H|^3\ dx \le C\|v\|^2_{L_s(B)}\|\tilde H\|^{3(1-\lambda)}_{L_3(B)}\|\nabla|\tilde H|^{\frac32}\|^{2\lambda}_{L_2(B)}$$

Applying the Young inequality $ab \le \varepsilon a^p + C_\varepsilon b^{p'}$ with $p = \frac{1}{\lambda}$ and $p' = \frac{1}{1-\lambda}$ we obtain

$$\int\limits_B |v|^2|\tilde H|^3\ dx \le \frac14\int\limits_B |\tilde H||\nabla\tilde H|^2\ dx + C\|v\|^{\frac{2}{1-\lambda}}_{L_s(B)}\|\tilde H\|^3_{L_3(B)}$$

Recalling that $\lambda = \frac{3}{s}$ and $1-\lambda = 1-\frac{3}{s} = \frac{2}{r}$ and absorbing the terms containing $\int |\tilde H||\nabla\tilde H|^2\ dx$ into the left-hand side of (2.2) we finally obtain

$$\frac13\frac{d}{dt}\|\tilde H\|^3_{L_3(B)} + \int\limits_B |\tilde H||\nabla\tilde H|^2\ dx \le C\|v\|^r_{L_s(B)}\|\tilde H\|^3_{L_3(B)} + C\int\limits_B |F||\tilde H|^2\ dx. \tag{2.3}$$

To finish the proof of Lemma we need to estimate $\int\limits_B |F||H|^2\ dx$. By Hölder inequality we get

$$\int\limits_B |F||\tilde H|^2\ dx \le \|F\|_{L_2(B)}\|\tilde H\|^2_{L_4(B)}.$$

Taking $\mu = \frac38$ so that $\frac14 = \frac{1-\mu}{3} + \frac{\mu}{9}$, using the interpolation inequality

$$\|\tilde H\|_{L_4(B)} \le C\|\tilde H\|^{1-\mu}_{L_3(B)}\|\tilde H\|^{\mu}_{L_9(B)} \le C\|\tilde H\|^{1-\mu}_{L_3(B)}\|\nabla|\tilde H|^{\frac32}\|^{\frac{2\mu}{3}}_{L_2(B)}$$

we get

$$\int\limits_B |F||\tilde H|^2\ dx \le \|F\|_{L_2(B)}\|\tilde H\|^{\frac54}_{L_3(B)}\|\nabla|\tilde H|^{\frac32}\|^{\frac12}_{L_2(B)}.$$

Applying the Young inequality $ab \le \varepsilon a^p + C_\varepsilon b^{p'}$ with $p = 4$ and $p' = \frac43$ we obtain

$$\int\limits_B |F||\tilde H|^2\ dx \le \frac14\int\limits_B |\tilde H||\nabla\tilde H|^2\ dx + C\|F\|^{\frac43}_{L_2(B)}\|\tilde H\|^{\frac53}_{L_3(B)}$$

Absorbing the term $\frac{1}{4}\int\limits_B |\tilde{H}||\nabla \tilde{H}|^2\,dx$ into the left hand side of (2.3) and using the estimate $\|F\|^{\frac{4}{3}}_{L_2(B)}\|\tilde{H}\|^{\frac{5}{3}}_{L_3(B)} \le C(\|F\|^2_{L_2(B)}+1)(\|\tilde{H}\|^3_{L_3(B)}+1)$ we obtain from (2.3) the inequality

$$\frac{1}{3}\frac{d}{dt}\|\tilde{H}\|^3_{L_3(B)} + \int\limits_B |\tilde{H}||\nabla \tilde{H}|^2\,dx \le C_0\Big(\|v\|^r_{L_s(B)} + \|F\|^2_{L_2(B)} + 1\Big)\|\tilde{H}\|^3_{L_3(B)} + C,$$

from which the result follows by the Gronwall lemma. Lemma 2.1 is proved.

Remark 2.1 *From Lemma 2.1 by interpolation $L_{3,\infty}(Q)\cap L_{9,3}(Q) \hookrightarrow L_{s_2,r_2}(Q)$ with any s_2, r_2 satisfying $\frac{3}{s_2}+\frac{2}{r_2}=1$ we conclude that H belongs to the LPS- class with any s_2, r_2:*

$$H \in L_{s_2,r_2}(Q), \qquad \frac{3}{s_2}+\frac{2}{r_2}=1, \quad s_2 \ge 3.$$

In particular, we can take $s_2 = r_3 = 5$ or $s_2 = s$, $r_2 = r$ where s and r are fixed in Theorem 1.3.

Lemma 2.2 *Assume v, $H \in L_{2,\infty}(Q)\cap W^{1,0}_2(Q)$, $p \in L_{\frac{3}{2}}(Q)$ satisfy the MHD system in Q:*

$$\left.\begin{array}{c} \partial_t v + (v\cdot\nabla)v - \Delta v + \nabla p = \mathrm{rot}\, H \times H \\ \mathrm{div}\, v = 0 \\ \partial_t H - \Delta H = \mathrm{rot}(v \times H) \\ \mathrm{div}\, H = 0 \end{array}\right\} \quad \text{in } Q. \tag{2.4}$$

Assume additionally that v and H satisfy the LPS conditions:

$$v \in L_{s_1,r_1}(Q), \qquad H \in L_{s_2,r_2}(Q),$$

where

$$\frac{3}{s_i}+\frac{2}{r_i}=1, \quad s_i > 3, \quad i=1,2.$$

Then v and H are Hölder continuous on $\bar{Q}(\frac{1}{2})$.

Proof: The proof of Lemma 2.2 is similar to the proof of the corresponding result for the Navier-Stokes system and the heat equation, see, for example, [11], [10] and reference there.

3 ε- Regularity Conditions

Existence of weak solutions (analogues to the Leray-Hopf solutions for the NSE) to various initial-boundary value problems for the MHD system was obtained by O.A. Ladyzhenskaya and V.A. Solonnikov in [5]. Originally these solutions provide no information about pressure. By small improvement of arguments of [5] (involving coercive estimates for the linearization of the MHD system) it is not difficult to prove existence of analogues of the so-called *suitable weak solutions* to the MHD system (the notion of suitable weak solution for the NSE equations was introduced by Scheffer [9], see also [1]).

Definition. The functions (v, H, p) are *the suitable weak solution to the MHD system in* $Q_T = \Omega \times (-T, 0)$ iff for any compact domain $\Omega' \Subset \Omega$ and any $T' < T$, $Q'_T = \Omega' \times (-T', 0)$

$$v \in L_{2,\infty}(Q'_T) \cap W^{1,0}_2(Q'_T), \qquad p \in L_{3/2}(Q'_T),$$

the equations (1.1), (1.2) are satisfied in Q_T in the sense of distributions, and, moreover, the following *Local Energy Inequality* holds

$$\begin{gathered}\int\limits_\Omega (|v(x,t)|^2 + |H(x,t)|^2)\,\zeta(x,t)\;dx + \int\limits_{-T}^{t}\int\limits_\Omega (|\nabla v|^2 + |\nabla H|^2)\,\zeta\;dxdt' \le \\ \le \int\limits_{Q_T} (|v|^2 + |H|^2)\,(\partial_t\zeta + \Delta\zeta)\;dxdt' + \\ + \int\limits_{Q_T} (|v|^2 + |H|^2 + 2p)\,v\cdot\nabla\zeta\;dxdt' - 2\int\limits_{Q_T} (v\cdot H)(v\cdot\nabla\zeta)\;dxdt',\end{gathered} \tag{3.1}$$

for a.e. $t \in (-T, 0)$ and all $\zeta \in C^\infty_0(\Omega \times (-T, 0])$, $\zeta \ge 0$.

One of the pleasant properties of suitable weak solutions is that they have locally integrable first time and second spatial derivatives, i.e.

$$v,\ H \in W^{2,1}_{\frac54,loc}(Q_T), \qquad p \in W^{1,0}_{\frac54,loc}(Q_T).$$

In particular, these means that v and $H \in C((-T, 0]; L_{\frac54,loc}(\Omega))$. If we assume additionally that v or H belongs to the space $L_\infty(-T, 0; L_q(\Omega))$ with some $q > \frac54$, then these function can be redefined on a set of moments of time of measure zero in such a way that v or H become continuous in time with values in $L_q(\Omega')$ equipped by the weak topology, i.e for any $w \in L_{q'}(\Omega')$ the function

$$t \mapsto \int\limits_{\Omega'} v(x,t)\cdot w(x)\;dx \quad \left(\text{or} \quad \int\limits_{\Omega'} H(x,t)\cdot w(x)\;dx\right) \quad \text{is continuous.}$$

In particular, this means that suitable weak solutions belonging to the class $L_\infty(-T, 0; L_q(\Omega))$ have values in $L_q(\Omega')$ for EVERY moment of time. Below we always assume that our suitable weak solutions have this property from the very beginning.

For suitable weak solutions to the MHD system it is possible to obtain the following ε-regularity theorem:

Theorem 3.1 *There is an absolute constant $\varepsilon_* > 0$ with the following property. For any suitable weak solution (v, p, H) of the system (1.1), (1.2) in $Q(z_0, R_0)$ if for some $R \leq R_0$*

$$\frac{1}{R^2} \int\limits_{Q(z_0,R)} \left(|v|^3 + |H|^3 + |p|^{\frac{3}{2}}\right) \, dxdt < \varepsilon_*,$$

then v and H are Hölder continuous on $\bar{Q}(z_0, \frac{R}{2})$.

Theorem 3.2 *There is an absolute constant $\varepsilon_0 > 0$ with the following property. Let (v, p, H) be a suitable weak solution of the system (1.1), (1.2) in $Q(z_0, R)$ and assume additionally that one of the following two condition holds:*

$$\text{either} \qquad \limsup_{\rho \to 0} \sup_{t \in (t_0 - \rho^2, t_0)} \frac{1}{\rho} \int\limits_{B(x_0,\rho)} \left(|v|^2 + |H|^2\right) \, dx < \varepsilon_0, \tag{3.2}$$

$$\text{or} \qquad \limsup_{\rho \to 0} \frac{1}{\rho} \int\limits_{Q(z_0,\rho)} \left(|\nabla v|^2 + |\nabla H|^2\right) \, dx < \varepsilon_0. \tag{3.3}$$

Then there is $\rho_* \leq R$ such that v and H are Hölder continuous on $\bar{Q}(z_0, \frac{\rho_*}{2})$.

Theorem 3.3 *Let (v, H, p) be a suitable weak solution of the MHD system in $Q(z_0, R)$ and assume additionally v, H are Hölder continuous on $Q(z_0, R)$. Then for any $k \in \mathbb{N}$ the functions $\nabla^k v$, $\nabla^k H$ are Hölder continuous on $Q(z_0, \frac{R}{2})$ and*

$$\sup_{z \in Q(z_0, \frac{R}{2})} \left(|\nabla^k v| + |\nabla^k H|\right) \leq \frac{C_k}{R^k}$$

The proofs of Theorem 3.1 — Theorem 3.3 can be obtained by the routine reproduction of the corresponding arguments for the NSE equations, see, for example, [1], [2], [4], [7].

4 Proof of Theorem 1.1

Obviously, it is sufficient to prove Hölder continuity of v and H near the origin. By contradiction, we assume that $(0, 0)$ is not a regular point for v, H. Then by Theorem 3.2 there is a sequence $R_k \to 0$ such that

$$\sup_{t \in (-R_k^2, 0)} \frac{1}{R_k} \int\limits_{B(R_k)} \left(|v|^2 + |H|^2\right) \, dx \geq \varepsilon_*. \tag{4.1}$$

Extend functions (v,p,H) outside Q by zero and denote by v^k, p^k and H^k the following functions:

$$v^k(y,s)=R_k\ v(R_ky,R_k^2s),\qquad H^k(y,s)=R_k\ H(R_ky,R_k^2s),$$
$$p^k(y,s)=R_k^2\ p(R_ky,R_k^2s).$$

The functions (v^k,H^k,p^k) satisfy the equations (1.1), (1.2) in $Q(1/R_k)$ and, moreover, thanks to (4.1)

$$\sup_{s\in(-1,0)}\int\limits_B\left(|v^k|^2+|H^k|^2\right)\ dy\geq\varepsilon_*. \tag{4.2}$$

It turns out that the sequence (v^k,H^k,p^k) has a subsequence that converges in the appropriate sense to some limit functions (v^0,H^0,p^0) which are the global solution to the MHD system (i.e. they satisfy the MHD system in $\mathbb{R}^3\times(-\infty,0)$):

Lemma 4.1 *There exist and a subsequence $(v^{k_j},H^{k_j},p^{k_j})$ and the functions (v^0,H^0,p^0) which are a suitable weak solution of the MHD equations (1.1), (1.2) in $\Pi\equiv\mathbb{R}^3\times(-\infty,0)$ such that*

$$v^0,\ H^0\in L_{3,\infty}(\Pi),\qquad p^0\in L_{\frac32,\infty}(\Pi), \tag{4.3}$$

$$v^{k_j}\overset{*}{\rightharpoonup}v^0\quad \textit{and}\qquad H^{k_j}\overset{*}{\rightharpoonup}H^0\quad \textit{in}\ \ L_{3,\infty}(\Pi), \tag{4.4}$$

and for any bounded domain $\Omega\Subset\mathbb{R}^3$ and any $T>0$

$$v^0\in W^{1,0}_2(\Omega\times(-T,0))\cap W^{2,1}_{\frac43}(\Omega\times(-T,0))\cap C([-T;0];L_2(\Omega)), \tag{4.5}$$

$$p^{k_j}\rightharpoonup p^0\quad \textit{in}\ L_{\frac32}(\Omega\times(-T,0)), \tag{4.6}$$

$$v^{k_j}\to v^0\quad \textit{and}\quad H^{k_j}\to H^0\quad \textit{in}\ \ C([-T;0];L_2(\Omega)), \tag{4.7}$$

$$v^{k_j}\to v^0\quad \textit{and}\qquad H^{k_j}\to H^0\quad \textit{in}\ \ L_3(\Omega\times(-T,0)). \tag{4.8}$$

Moreover, the functions (v^0,H^0,p^0) have the following property:

$$v^0(y,0)=0,\qquad H^0(y,0)=0,\qquad \textit{for a.e.}\ \ y\in\mathbb{R}^3. \tag{4.9}$$

The proof of Lemma 4.1 essentially repeats the arguments of the paper [2]. We present the proof Lemma 4.1 below just for the reader's convenience.

Convergence (4.7) makes it possible to pass to the limit in the inequality (4.2). So we obtain the bound from below

$$\sup_{s\in(-1,0)}\int\limits_B\left(|v^0|^2+|H^0|^2\right)\ dy\geq\varepsilon_*. \tag{4.10}$$

The principal result of our work is the following lemma:

Lemma 4.2 *Assume functions (v^0, H^0, p^0) are a suitable weak solution of the MHD system in $\Pi_{T_0} \equiv \mathbb{R}^3 \times (-T_0, 0)$, $T_0 > 2$, and assume conditions (4.3), (4.5), (4.9) hold. Then*

$$v^0 \equiv 0, \qquad H^0 \equiv 0 \qquad \textit{a.e. in} \quad \Pi_{T_0-1}.$$

As soon as Lemma 4.2 is established, it provides the contradiction with (4.10). Hence the result of Theorem 1.1 follows.

Proof of Lemma 4.1. From the obvious estimate

$$\|v^k\|_{L_{3,\infty}(\Pi)} + \|H^k\|_{L_{3,\infty}(\Pi)} \le \|v\|_{L_{3,\infty}(Q)} + \|H\|_{L_{3,\infty}(Q)} \le C \tag{4.11}$$

we obtain convergence (4.4) as well as the first two inclusions in (4.3). Taking divergence of the equation (1.1) and taking into account the identity

$$\operatorname{rot} H \times H = (H \cdot \nabla)H - \frac{1}{2}\nabla|H|^2 \tag{4.12}$$

we obtain the relation for pressure

$$\Delta p = \operatorname{div}\operatorname{div}(H \otimes H - v \otimes v) - \frac{1}{2}\Delta|H|^2.$$

We split pressure as

$$p = p_1 + p_2,$$

where p_1 is defined as the solution to the following boundary-value problem:

$$\int\limits_B p_1 \Delta\psi \, dx = \int\limits_B \Big((H \otimes H - v \otimes v) : \nabla^2\psi - \frac{1}{2}|H|^2\Delta\psi \Big) \, dx,$$

for a.e. $t \in (-1, 0)$ and all $\psi \in W^2_3(B) \cap \overset{\circ}{W}{}^1_3(B)$. By an appropriate choice of the test function ψ we obtain the following estimate for p_1:

$$\|p_1\|_{L_{\frac{3}{2},\infty}(Q)} \le C\Big(\|H\|^2_{L_{3,\infty}(Q)} + \|v\|^2_{L_{3,\infty}(Q)} \Big).$$

On the other hand, for a.e. $t \in (-1, 0)$ the function $p_2(\cdot, t)$ is harmonic. Hence $p_2 \in L_{\frac{3}{2}}(-1, 0, L^\infty(B(\frac{5}{6})))$ and $\|p_2(\cdot, t)\|_{L_\infty(B(\frac{5}{6}))} \le c\|p_2(\cdot, t)\|_{L_{\frac{3}{2}}(B)}$. So, we obtain the estimate for p_2:

$$\|p_2\|_{L_{\frac{3}{2}}(-1,0,L^\infty(B(\frac{5}{6})))} \le C\left(\|v\|^2_{L_{3,\infty}(Q)} + \|H\|^2_{L_{3,\infty}(Q)} + \|p\|_{L_{\frac{3}{2}}(Q)} \right).$$

Splitting now the functions p^k in the same way, $p^k = p^k_1 + p^k_2$, where $p^k_j(y, s) = R^2_k p_j(R_k y, R^2_k s)$, $j = 1, 2$, for any $\Omega \Subset \mathbb{R}^3$ we obtain estimates

$$\|p^k_1\|_{L_{\frac{3}{2},\infty}(\Pi)} \le \|p_1\|_{L_{\frac{3}{2},\infty}(Q)} \le C,$$

$$\|p_2^k\|_{L_{\frac{3}{2}}(-\infty,0;L_\infty(\Omega))} \leq R_k \|p_2\|_{L_{\frac{3}{2}}(-1,0;L_\infty(B(\frac{5}{6})))} \to 0, \qquad \text{as } R_k \to 0.$$

From the estimate for p_1^k we obtain existence of $p^0 \in L_{\frac{3}{2},\infty}(\Pi)$ such that $p_1^k \overset{*}{\rightharpoonup} p^0$ in $L_{\frac{3}{2},\infty}(\Pi)$. Taking into account the estimate for p_2^k we obtain (4.6).

Let us prove now inclusions (4.5). Take an arbitrary $\zeta \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$ and consider $\zeta^k(x,t) = \zeta(\frac{x}{R^k}, \frac{t}{R_k^2})$. Choose k large enough so that $\operatorname{supp}\zeta^k \Subset B \times (-1,1)$. Substituting ζ^k into the Local Energy Inequality (3.1) and making change of variables $y = \frac{x}{R_k}$, $s = \frac{t}{R_k^2}$, we arrive at the relation

$$\begin{gathered}
\int\limits_{\mathbb{R}^3} \left(|v^k(y,s)|^2 + |H^k(y,s)|^2\right) \zeta(y,s)\, dy + \int\limits_{-\infty}^{s} \int\limits_{\mathbb{R}^3} \left(|\nabla_y v^k|^2 + |\nabla_y H^k|^2\right) \zeta \, dyds \leq \\
\leq \int\limits_{\Pi} \left(|v^k|^2 + |H^k|^2\right) (\partial_s \zeta + \Delta_y \zeta)\, dyds + \\
+ \int\limits_{\Pi} \left(|v^k|^2 + |H^k|^2 + 2p^k\right) v^k \cdot \nabla_y \zeta \, dyds + \\
-2 \int\limits_{\Pi} (v^k \cdot H^k)(v^k \cdot \nabla_y \zeta)\, dyds,
\end{gathered} \tag{4.13}$$

that holds for a.e. $s < 0$, for any $\zeta \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$ and for k large enough. It is easy to see that the RHS of (4.13) can be estimated by L_3- norms of v^k, H^k, and $L_{\frac{3}{2}}$- norm of p^k. Hence taking an arbitrary $\Omega \Subset \mathbb{R}^3$ and $T > 0$ after appropriate choice of ζ we obtain the estimate

$$\|v^k\|_{W_2^{1,0}(\Omega\times(-T,0))} + \|H^k\|_{W_2^{1,0}(\Omega\times(-T,0))} \leq C, \tag{4.14}$$

from which the first inclusion in (4.5) follows. The estimates (4.14), (4.11), and the interpolation result $L_{3,\infty}(Q_T) \cap W_2^{1,0}(Q_T) \hookrightarrow L_4(Q_T)$ provide boundedness of v^k and H^k in $L_4(\Omega \times (-T,0))$. By Hölder inequality we obtain that all bilinear terms

$$\operatorname{rot} H^k \times H^k, \quad (v^k \cdot \nabla) v^k, \quad \operatorname{rot}(v^k \times H^k) \equiv (H^k \cdot \nabla) v^k - (v^k \cdot \nabla) H^k$$

are bounded in $L_{\frac{4}{3}}(\Omega \times (-T,0))$. Considering bilinear terms as given external forces and applying coercive estimates for the linear Stokes and the heat equations, for an arbitrary $\Omega' \Subset \Omega$ and $T' \in (0,T)$ we derive the estimate

$$\|v^k\|_{W_{\frac{4}{3}}^{2,1}(\Omega'\times(-T',0))} + \|H^k\|_{W_{\frac{4}{3}}^{2,1}(\Omega'\times(-T',0))} + \|\nabla p^k\|_{L_{\frac{4}{3}}(\Omega'\times(-T',0))} \leq C, \tag{4.15}$$

from which the second assertion in (4.5) follows. (To simplify our notations below we will omit prime and we will use notation Ω and $(-T,0)$ for an arbitrary domain and arbitrary time interval, passing to smaller domains when necessary, but keeping the same notations for them).

In particular, from (4.15) and the imbedding

$$W_{\frac{4}{3}}^{2,1}(\Omega \times (-T,0)) \overset{\text{comp}}{\hookrightarrow} C([-T,0]; L_{\frac{4}{3}}(\Omega))$$

we obtain strong convergence $v^k \to v^0$ and $H^k \to H^0$ in $C([-T,0];L_{\frac{4}{3}}(\Omega))$. In [2] it was shown that the following imbedding is true:

$$C([-T;0];L_{\frac{4}{3}}(\Omega))\cap L_\infty(-T,0;L_3(\Omega))\hookrightarrow C([-T;0];L_2(\Omega))$$

$$\|w\|_{C([-T,0];L_2(\Omega))}\le c\|w\|^{\frac{2}{5}}_{C([-T,0];L_{\frac{4}{3}}(\Omega))}\|w\|^{\frac{3}{5}}_{L_\infty(-T,0;L_3(\Omega))}$$

and hence v^k, H^k, v^0, $H^0\in C([-T,0];L_2(\Omega))$. Moreover, from the relation

$$\begin{gathered}\|v^k-v^0\|_{C([-T,0];L_2(\Omega))}\le c\|v^k-v^0\|^{\frac{2}{5}}_{C([-T,0];L_{\frac{4}{3}}(\Omega))}\times\\ \times\Big(\|v^k\|^{\frac{3}{5}}_{L_{3,\infty}(\Omega\times(-T,0))}+\|v^0\|^{\frac{3}{5}}_{L_{3,\infty}(\Omega\times(-T,0))}\Big)\end{gathered}$$

the last assertion in (4.5) and convergence (4.6) follow. Finally, convergence (4.8) follows from the fact that the sequences v^k and H^k are bounded in $L_4(\Omega\times(-T,0))$ and precompact in $L_2(\Omega\times(-T,0))$. Using (4.8) we can pass to the limit in (4.13) and in the MHD equations for (v^k,H^k,p^k). So, we conclude that the limit functions (v^0,H^0,p^0) are a suitable weak solution of the MHD equations in Π.

Let us prove (4.9). For any $x_*\in\mathbb{R}^3$ we have

$$\begin{gathered}\|v^0(\cdot,0)\|_{L_2(B(x_*,1))}\le\|v^k(\cdot,0)-v^0(\cdot,0)\|_{L_2(B(x_*,1))}+\|v^k(\cdot,0)\|_{L_2(B(x_*,1))}\le\\ \le\|v^k-v^0\|_{C([-T,0];L_2(B(x_*,1))}+R_k^{-1/2}\|v(\cdot,0)\|_{L_2(B(R_kx_*,R_k))}\end{gathered}$$

For the last term due to Hölder inequality we have the following estimate:

$$R_k^{-1/2}\|v(\cdot,0)\|_{L_2(B(R_kx_*,R_k))}\le C\|v(\cdot,0)\|_{L_3(B(R_kx_*,R_k))}\to 0\quad\text{as } R_k\to 0.$$

(We remind here that we assume $v(\cdot,t)\in L_3(B)$ for every $t\in[-T,0]$). So, the first assertion in (4.9) follows from (4.7) and absolute continuity of Lebegue integral. The identity (4.9) for H^0 follows in the same way. Lemma 4.1 is proved.

Proof of Lemma 4.1, Part I. Let $T=T_0-1$. From Theorem 3.1 we conclude that there exist a cylinder $Q_{R_1,T}\equiv B(R_1)\times(-T;0)$ such that all spatial derivatives of v^0 and H^0 are bounded and Hölder continuous on $\Pi_T\setminus Q_{R_1,T}$. Let us fix some of these bounds:

$$\|v^0\|_{L_\infty(\Pi_T\setminus Q_{R_1,T})}+\|H^0\|_{L_\infty(\Pi_T\setminus Q_{R_1,T})}\le M_0,$$

$$\|\nabla v^0\|_{L_\infty(\Pi_T\setminus Q_{R_1,T})}+\|\nabla H^0\|_{L_\infty(\Pi_T\setminus Q_{R_1,T})}\le M_1,$$

$$\|\nabla^2 v^0\|_{L_\infty(\Pi_T\setminus Q_{R_1,T})}+\|\nabla^2 H^0\|_{L_\infty(\Pi_T\setminus Q_{R_1,T})}\le M_2.$$

Moreover, for a.e. $t\in(-T,0)$ the pressure $p^0(\cdot,t)\in L_{3/2}(\mathbb{R}^3)$ satisfies the identity

$$\Delta p^0=\operatorname{div}\operatorname{div}(H^0\otimes H^0-v^0\otimes v^0)-\frac{1}{2}\Delta|H^0|^2.$$

As the norm $\|p^0(\cdot,t)\|_{L_{3/2}(\mathbb{R}^3)}$ is bounded with respect to $t \in (-T,0)$ for any $k \in \mathbb{N}$ the following assertion holds:

$$\nabla^k p^0, \quad \partial_t \nabla^{k-1} v^0 \quad \text{ are locally bounded in } \ \Pi_T \setminus Q_{R_1,T}. \tag{4.16}$$

Our basic tools are the following propositions on the unique continuation for the heat operator, see [2], [3]:

Proposition 4.1 *Assume there exist $R>0$, $T>0$ such that $W \in W^{2,1}_{2,\text{loc}}(\Pi_T \setminus Q_{R,T};\mathbb{R}^k)$ satisfies the following conditions*

$$\exists\ M>0: \qquad |\partial_t W - \Delta W| \le M\big(|W| + |\nabla W|\big) \quad \textit{a.e. in } \ \Pi_T \setminus Q_{R,T},$$

$$\exists\ C>0: \qquad |W(x,t)| \le e^{C|x|^2} \quad \textit{a.e. in } \ \Pi_T \setminus Q_{R,T},$$

$$W(\cdot,0)=0 \qquad \textit{a.e. in } \ \mathbb{R}^n \setminus B_R.$$

Then $W \equiv 0$ a.e. in $\Pi_T \setminus Q_{R,T}$.

Proposition 4.2 *Assume there exist $R>0$, $T>0$ such that $W \in W^{2,1}_{2,\text{loc}}(Q_{R,T};\mathbb{R}^k)$ satisfies the inequality*

$$\exists\ M>0: \qquad |\partial_t W - \Delta W| \le M\big(|W| + |\nabla W|\big) \quad \textit{a.e. in } \ Q_{R,T}.$$

Assume also that for some cylinder $\tilde{Q} \equiv B(x_0,\rho)\times(t_1,t_2) \subset Q_{R,T}$

$$W \equiv 0 \qquad \textit{a.e. in } \ \tilde{Q}.$$

Then $W \equiv 0$ a.e. in $B(R)\times(t_1,t_2)$.

While the magnetic field satisfies the parabolic equation

$$\partial_t H^0 - \Delta H^0 = (H^0\cdot\nabla)v^0 - (v^0\cdot\nabla)H^0.$$

from the very beginning, for the velocity filed v^0 we have only the Stokes-type system, for which to the best of our knowledge the results on the unique continuation similar to Propositions 4.1, 4.2 are unknown. So, to eliminate the pressure and to get a parabolic equation we take rot of (1.1) and obtain the system for $\omega^0 \equiv \operatorname{rot} v^0$. Our main goal is to control the structure of weak terms we obtain under this operation. We denote by $J^0 = \operatorname{rot} H^0$, we also use representation (4.12) and the relation $\operatorname{rot}(v^0\times H^0) = (H^0\cdot\nabla)v^0 - (v^0\cdot\nabla)H^0$:

$$\partial_t\omega^0 - \Delta\omega^0 = (\omega^0\cdot\nabla)v^0 - (v^0\cdot\nabla)\omega^0 + (H^0\cdot\nabla)J^0 - (J^0\cdot\nabla)H^0.$$

As the RHS of this equation includes the term $(H^0 \cdot \nabla)J^0$ which contains the second derivatives of H^0 we have to differentiate also the magnetic equation:

$$\partial_t H^0_{,k} - \Delta H^0_{,k} = (H^0_{,k} \cdot \nabla)v^0 + (H^0 \cdot \nabla)v^0_{,k} - (v^0_{,k} \cdot \nabla)H^0 - (v^0 \cdot \nabla)H^0_{,k}$$

Let us introduce the new 15-component vector $W = (H^0, \omega^0, H^0_{,1}, H^0_{,2}, H^0_{,3})$ and also (to make our considerations more observable) we introduce three auxiliary vector functions consisting of some components of W: the 3-component vector $W^{(0)} = H^0$, the 3-component vector $W^{(1)} = \omega^0$ and the 9-component vector $W^{(2)} = (H^0_{,1}, H^0_{,2}, H^0_{,3})$

Thanks to our equations it is easy to see that this vector-functions satisfy the differential inequalities in $\Pi_T \setminus Q_{R_1,T}$:

$$|\partial_t W^{(0)} - \Delta W^{(0)}| \leq M_1 |W^{(0)}| + M_0 |W^{(1)}|,$$

$$|\partial_t W^{(1)} - \Delta W^{(1)}| \leq M_1 |W^{(1)}| + M_0 |\nabla W^{(1)}| + M_2 |W^{(0)}| + M_1 |W^{(2)}|,$$

$$|\partial_t W^{(2)} - \Delta W^{(2)}| \leq M_1 |W^{(2)}| + M_2 |W^{(0)}| + M_1 |W^{(2)}| + M_0 |\nabla W^{(2)}|,$$

Combining these three inequalities we arrive at the differential inequality for $W = (W^{(0)}, W^{(1)}, W^{(2)})$:

$$|\partial_t W - \Delta W| \leq C_{M_0,M_1,M_2} \big(|W| + |\nabla W| \big) \quad \text{in} \quad \Pi_T \setminus Q_{R_1,T}.$$

Moreover, as identities (4.9) hold and the functions $v^0(\cdot, 0)$ and $H^0(\cdot, 0)$ are smooth on $\mathbb{R}^3 \setminus B_{R_1}$, differentiating these relations we arrive at

$$W(x, 0) = 0 \qquad \text{for } x \in \mathbb{R}^3 \setminus B_{R_1}.$$

Applying Proposition 4.1 we obtain the identity

$$W \equiv 0 \quad \text{in} \quad \Pi_T \setminus Q_{R_1,T}.$$

Proof of Lemma 4.1, Part II.

At this step of the proof we already know that the functions v^0, H^0 possess the following properties:

- $\operatorname{rot} v^0 = 0$, $\operatorname{div} v^0 = 0$, and hence $\Delta v^0 = 0$ on $\Pi_T \setminus Q_{R_1,T}$.
- $H^0 \equiv 0$ on $\Pi_T \setminus Q_{R_1,T}$.
- $\nabla^k v^0$ are bounded and Hölder continuous on $\Pi_T \setminus Q_{R_1,T}$ for any $k \in \mathbb{N}$.
- v^0, $H^0 \in L_{3,\infty}(\Pi_T)$, $p^0 \in L_{\frac{3}{2},\infty}(\Pi_T)$ satisfy the MHD system in Π_T.

So, roughly speaking, we have shown at this step that for every moment of time our function W has a compact support. Our next goal is to show that W must vanish identically in Π_T. Unfortunately, due to the lack of smoothness of the original functions v^0 and H^0, the function W is not even well-defined inside $Q_{R_1,T}$. We are going to show that for almost every $t_0 \in (-T, 0)$ there is a strip $(t_0, t_0 + \delta_0)$ with some $\delta_0 = \delta_0(t_0) > 0$ such that v^0 and H^0 are sufficiently smooth (and hence W is well-defined) in this strip.
Take $R_2 = R_1 + 1$ and $R_3 = R_2 + 1$. Consider the cut-off function $\zeta \in C_0^\infty(B_{R_3})$ such that $\zeta \equiv 1$ on B_{R_2}. We introduce functions

$$u = \zeta v^0, \qquad q = \zeta p^0, \qquad H = H^0.$$

(We have redenote here H^0 by H just to simplify our notation). It is easy to see that (u, H, q) satisfy the identities:

$$\left.\begin{array}{c} \partial_t u - \Delta u + \operatorname{div}(u \otimes u) + \nabla q = \operatorname{rot} H \times H + F \\ \operatorname{div} u = v^0 \cdot \nabla \zeta \end{array}\right\} \quad \text{in} \quad \Pi_T,$$

$$\left.\begin{array}{c} \partial_t H - \Delta H = \operatorname{rot}(u \times H) \\ \operatorname{div} H = 0 \end{array}\right\} \quad \text{in} \quad \Pi_T,$$

$$u|_{\partial B_{R_3}} = 0, \qquad H_\nu|_{\partial B_{R_3}} = 0, \qquad (\operatorname{rot} H)_{tan}|_{\partial B_{R_3}} = 0,$$

where ν is the outer normal to ∂B_{R_3}, $H_\nu = H \cdot \nu$, $(\operatorname{rot} H)_{tan} = \operatorname{rot} H \times \nu$,

$$F = -2(\nabla v^0)\nabla\zeta - v^0\Delta\zeta + (\zeta^2 - \zeta)\operatorname{div}(v^0 \otimes v^0) + (v^0 \otimes v^0)\nabla\zeta + p^0\nabla\zeta.$$

We remark that thanks to the condition $\operatorname{supp} H \subset \bar{Q}_{R_1,T}$ we have the identities $\zeta \operatorname{rot} H \times H = \operatorname{rot} H \times H$ and $(v^0 \times H) \times \nabla\zeta = 0$. Note also that the function F and all its spatial derivatives are bounded on Π_T.
To improve the divergence-free conditions for u we introduce for a.e. $t \in (-T, 0)$ the functions $(w(\cdot, t), r(\cdot, t))$ which are the solution to Stokes-type problem:

$$\begin{array}{c} -\Delta w + \nabla r = 0, \quad \operatorname{div} w = g \equiv v^0 \cdot \nabla\zeta \quad \text{in} \quad B_{R_3}, \\ w|_{\partial B_{R_3}} = 0. \end{array} \tag{4.17}$$

From the elliptic theory it is well-known (see [12]) that the functions (w, r) satisfy the estimate for any $k \geq 0$:

$$\|w\|_{W_2^{k+2}(B_{R_3})} + \|r\|_{W_2^{k+1}(B_{R_3})} \leq C\|g\|_{W_2^{k+1}(B_{R_3})}.$$

Taking the derivative of the equations (4.17) with respect to t we get the analogues estimate for $\partial_t w$:

$$\|\partial_t w\|_{W_2^{k+2}(B_{R_3})} + \|\partial_t r\|_{W_2^{k+1}(B_{R_3})} \leq C\|\partial_t g\|_{W_2^{k+1}(B_{R_3})}.$$

Taking into account (4.16) and the fact that the function $g \equiv v^0 \cdot \nabla\zeta$ is supported on $Q_{R_3,T} \setminus Q_{R_2,T}$ we conclude that for any $k \in \mathbb{N}$

$$\partial_t \nabla^{k-1} w \ \in \ L_\infty(Q_{R_3,T}), \qquad \nabla^k w \ \in \ C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_{R_3,T}). \tag{4.18}$$

Let us represent u and q as the sums

$$u = U + w, \qquad q = P + r$$

where U, P and H satisfy the relations

$$\left.\begin{array}{c} \partial_t U - \Delta U + \operatorname{div}(U \otimes U) + \operatorname{div}(U \otimes w) + \\ + \operatorname{div}(w \otimes U) + \nabla P = \tilde{F} + \operatorname{rot} H \times H \\ \operatorname{div} U = 0 \end{array}\right\} \quad \text{in} \quad Q_{R_3,T}, \tag{4.19}$$

$$\left.\begin{array}{c} \partial_t H - \Delta H - \operatorname{rot}(w \times H) = \operatorname{rot}(U \times H) \\ \operatorname{div} H = 0 \end{array}\right\} \quad \text{in} \quad Q_{R_3,T}, \tag{4.20}$$

$$U|_{\partial B_{R_3}} = 0, \qquad H_\nu|_{\partial B_{R_3}} = 0, \qquad (\operatorname{rot} H)_{tan}|_{\partial B_{R_3}} = 0, \tag{4.21}$$

where

$$\tilde{F} = F - \partial_t w - \operatorname{div}(w \otimes w).$$

As U, $H \in L_2(-T, 0; W_2^1(B_{R_3}))$ hence there is a set $E \subset [-T, 0]$ such that $\operatorname{meas}\{[-T, 0] \setminus E\} = 0$ and for any $t_0 \in E$ the inclusions $U(\cdot, t_0)$, $H(\cdot, t_0) \in W_2^1(B_{R_3})$ hold. Let us fix $t_0 \in E$. Note also that thanks to (4.18) the function $\tilde{F}$ and the coefficient w in the system (4.19), (4.20) are bounded in $Q_{R_3,T}$ together with all their spatial derivatives.
Due to the well-posedness of the system (4.19), (4.20), (4.21) for the initial data $U(t_0) \in \overset{\circ}{J}{}_2^1(B_{R_3})$ and $H(t_0) \in J_{2,\nu}^1(B_{R_3})$ (see [5]) we conclude that for any $t_0 \in E$ there is $\delta = \delta(t_0) > 0$ such that

$$U, \quad H \ \in \ W_2^{2,1}(B_{R_3} \times (t_0, t_0 + \delta)).$$

Hence for any $k \in \mathbb{N}$ the functions $\nabla^k U$, $\nabla^k H$ are locally Hölder continuous on an open strip $\mathbb{R}^3 \times (t_0, t_0 + \delta)$ and thanks to (4.18) the same is true for $\nabla^k v$. Now we can take rot of the NSE equation and obtain the differential inequality

$$|\partial_t W - \Delta W| \le M_* \left(|W| + |\nabla W|\right)$$

for the 15-component vector function W introduced in Part I. This inequality holds in the strip $\mathbb{R}^3 \times [t_1, t_1 + \delta_1]$ for any $(t_1, t_1 + \delta_1) \Subset (t_0, t_0 + \delta)$ with the constant M_* depending on t_1, δ_1. Taking into account the identity $W \equiv 0$ a.e. in $\Pi_T \setminus Q_{R_1,T}$ and applying Proposition 4.2 we conclude that $W \equiv 0$ on $\mathbb{R}^3 \times [t_1, t_1 + \delta_1]$. Hence for every $t \in [t_1, t_1 + \delta_1]$ the function $v(\cdot, t)$ is harmonic in $\mathbb{R}^3$ and, moreover, it belongs to $L_3(\mathbb{R}^3)$. So we conclude that the function v vanishes identically on $[t_1, t_1 + \delta_1]$. As $t_1 \in (t_0, t_0 + \delta)$ is arbitrary due to the strong continuity of v with values in $L_{\frac{4}{3}}(B_{R_3})$ we obtain $v(\cdot, t_0) \equiv 0$. This implies that $v(\cdot, t_0) = 0$ for any $t_0 \in E$. Lemma 4.2 is proved.

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