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The role of oscillations in the global wellposedness of the 3-D incompressible anisotropic Navier-Stokes equations

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Abstract

Corresponding to the wellposedness result [2] for the classical 3-D Navier-Stokes equations (NS_ν) with initial data in the scaling invariant Besov space, $\mathcal{B}_{p,\infty}^{-1+\frac{3}{p}}$, here we consider a similar problem for the 3-D anisotropic Navier-Stokes equations (ANS_ν), where the vertical viscosity is zero. In order to do so, we first introduce the Besov-Sobolev type spaces, $\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}$ and $\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)$. Then with initial data in the scaling invariant space $\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}$, we prove the global wellposedness for (ANS_ν) provided the norm of initial data is small enough compared to the horizontal viscosity. In particular, this result implies the global wellposedness of (ANS_ν) with high oscillatory initial data (1.2).

1 Introduction

1.1 Introduction to the anisotropic Navier-Stokes equations

Let us first recall the classical (isotropic) Navier-Stokes system for incompressible fluids in the whole space:

$$(NS_\nu)\quad \begin{cases} \partial_t u + u\cdot\nabla u - \nu\Delta u = -\nabla p, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

where $u(t,x)$ denote the velocity, $p(t,x)$ the pressure and $x=(x_h,x_3)$ a point of $\mathbf{R}^3=\mathbf{R}^2\times\mathbf{R}$.

In this text, we are going to study a version of the system (NS_ν) where the usual Laplacian is substituted by the Laplacian in the horizontal variables, namely

$$(ANS_\nu)\quad \begin{cases} \partial_t u + u\cdot\nabla u - \nu\Delta_h u = -\nabla p, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases}$$

Systems of this type appear in geophysical fluids (see for instance [5]). It has been studied first by J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier in [6] and D. Iftimie in [10] where it is proved that (ANS_ν) is locally wellposed for initial data in the anisotropic space

$$H^{0,\frac{1}{2}+\varepsilon} \stackrel{\text{def}}{=} \Big\{u \in L^2(\mathbf{R}^3) \ / \ \|u\|^2_{\dot{H}^{\frac{1}{2}+\varepsilon}} \stackrel{\text{def}}{=} \int_{\mathbf{R}^3} |\xi_3|^{1+2\varepsilon} |\hat{u}(\xi_h,\xi_3)|^2 d\xi < +\infty\Big\},$$

for some $\varepsilon > 0$. Moreover, it is also proved that small enough data u_0 is in the sense that

$$\|u_0\|^{\varepsilon}_{L^2} \|u_0\|^{1-\varepsilon}_{\dot{H}^{0,\frac{1}{2}+\varepsilon}} \leq c\nu \tag{1.1}$$

for some sufficiently small constant c, then we have a global wellposedness result. Let us notice that the space in which uniqueness is proved is the space of continuous functions with value in $H^{0,\frac{1}{2}+\varepsilon}$ and the horizontal gradient of which belongs to $L^2([0,T]; H^{0,\frac{1}{2}+\varepsilon})$.

Let us observe that, as classical Navier-Stokes system, the system (ANS_ν) has a scaling. Indeed, if u is a solution of (ANS_ν) on a time interval $[0,T]$ with initial data u_0, then the vector field u_λ defined by $u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x)$ is also a solution of (ANS_ν) on the time interval $[0, \lambda^{-2}T]$ with the initial data $\lambda u_0(\lambda x)$. The smallness condition (1.1) is of course scaling invariant. But the norm $\|\cdot\|_{\dot{H}^{0,\frac{1}{2}+\varepsilon}}$ is not and this norm determines the level of regularity required to have wellposedness.

For classical Navier-Stokes system a lot of results of global wellposedness in scaling invariant space are available. The first one is the theorem of Fujita-Kato (see [9]) in which it is proved that the system (NS_ν) is globally wellposed for small initial data in the Sobolev space $\dot{H}^{\frac{1}{2}}$ which is the space of tempered distributions u such that

$$\|u\|^2_{\dot{H}^{\frac{1}{2}}} \stackrel{\text{def}}{=} \int_{\mathbf{R}^3} |\xi|\, |\widehat{u}(\xi)|^2 d\xi < \infty.$$

M. Paicu proved in [12] a theorem of the same type for the system (ANS_ν) in the case when the initial data u_0 belongs to the scaling invariant space $\mathcal{B}^{0,\frac{1}{2}}$ (see Definition 1.2 below).

On the other hand, the classical isotropic system (NS_ν), is globally wellposed for small initial data in Besov norms of negative index. Let us first recall the definition of the Besov norms of negative index.

Definition 1.1 *Let f be in $\mathcal{S}'(\mathbf{R}^3)$. Then we state, for positive s, and for (p,q) in $[1,\infty]^2$,*

$$\|f\|_{\dot{B}^{-s}_{p,q}} \stackrel{\text{def}}{=} \Big\| \|t^{\frac{s}{2}} e^{t\Delta} f\|_{L^p} \Big\|_{L^q(\mathbf{R}^+, \frac{dt}{t})}.$$

In [2], M. Cannone, Y. Meyer and F. Planchon proved that, if the initial data satisfies, for some p greater than 3, $\|u_0\|_{\dot{B}^{-1+\frac{3}{p}}_{p,\infty}} \leq c\nu$ for some constant c small enough, then the incompressible Navier-Stokes system is globally wellposed. Let us mention that H. Koch and D. Tataru generalized this theorem to the ∂BMO norm (see [11]).

In particular, this theorem implies that, for any function ϕ in the Schwartz space $\mathcal{S}(\mathbf{R}^3)$, if we consider the family of initial data u_0^ε defined by

$$u_0^\varepsilon(x) = \sin\Big(\frac{x_1}{\varepsilon}\Big) \left(0, -\partial_3\phi, \partial_2\phi, \right), \tag{1.2}$$

the system (NS_ν) is globally wellposed for such initial data when ε is small enough. The goal here is to prove the same for the anisotropic Navier-Stokes system (ANS_ν).

1.2 Statement of the results

Let us begin by the definition of the spaces we are going to work with. It requires an anisotropic version of dyadic decomposition of the Fourier space, let us first recall the following operators of localization in Fourier space, for $(k,\ell)\in\mathbf{Z}^2$,

$$\begin{aligned}\Delta_k^h a = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_h|)\widehat{a}), \quad &\text{and} \quad \Delta_\ell^v a = \mathcal{F}^{-1}(\varphi(2^{-\ell}|\xi_3|)\widehat{a}),\\ S_k^h a = \sum_{k'\leq k-1}\Delta_{k'}^h a, \quad &\text{and} \quad S_\ell^v a = \sum_{\ell'\leq \ell-1}\Delta_{\ell'}^v a,\end{aligned} \tag{1.3}$$

where $\mathcal{F}a$ and $\widehat{a}$ denote the Fourier transform of a, and φ a function in $\mathcal{D}\Big(\Big[\frac{3}{4},\frac{8}{3}\Big]\Big)$ such that

$$\forall \tau>0\,,\ \sum_{j\in\mathbf{Z}}\varphi(2^{-j}\tau)=1.$$

Before we present the space we are going to work with, let us first recall the Besov-Sobolev type space $\mathcal{B}^{0,\frac12}$ defined by M. Paicu in [12].

Definition 1.2 *We denote by $\mathcal{B}^{0,\frac12}$ the space of a in $\mathcal{S}(\mathbf{R}^3)$ such that*

$$\widehat{a}\in L^1_{loc} \quad \textit{and} \quad \|a\|_{\mathcal{B}^{0,\frac12}} \stackrel{def}{=} \sum_{\ell\in\mathbf{Z}} 2^{\frac{\ell}{2}}\|\Delta_\ell^v a\|_{L^2(\mathbf{R}^3)}<\infty.$$

In [12], M. Paicu proved the global wellposedness of (ANS_ν) for small initial data in $\mathcal{B}^{0,\frac12}$. In order to state Paicu's Theorem, let us introduce the following space.

Definition 1.3 *We denote by $\mathcal{B}^{0,\frac12}(T)$ the space of a in $C^\infty([0,T],\mathcal{B}^{0,\frac12}$ such that*

$$\|a\|_{\mathcal{B}^{0,\frac12}(T)} \stackrel{def}{=} \sum_{\ell\in\mathbf{Z}} 2^{\frac{\ell}{2}}\left(\|\Delta_\ell^v a\|_{L^\infty_T(L^2(\mathbf{R}^3))}+\nu^{\frac12}\|\nabla_h\Delta_\ell^v a\|_{L^2_T(L^2(\mathbf{R}^3))}\right)<\infty.$$

Now let us recall M. Paicu's theorem.

Theorem 1.1 *If $u_0\in\mathcal{B}^{0,\frac12}$, then a positive time T exists such that the system (ANS_ν) has a unique solution u in $\mathcal{B}^{0,\frac12}(T)$. Moreover, a constant c exists such that*

$$\|u_0\|_{\mathcal{B}^{0,\frac12}}\leq c\nu \Longrightarrow T=+\infty.$$

Let us note that u_0^ε defined in (1.2) is not small in this space $\mathcal{B}^{0,\frac12}$ no matter how small the parameter ε is. Our motivation to introduce the following spaces is to find a scaling invariant space such that in particular u_0^ε is small in this space for ε sufficient small.

Definition 1.4 *We denote by $\mathcal{B}_4^{-\frac12,\frac12}$ the space of a in $\mathcal{S}'(\mathbf{R}^3)$ such that*

$$\widehat{a}\in L^1_{loc} \quad \textit{and} \quad \|a\|_{\mathcal{B}_4^{-\frac12,\frac12}} \stackrel{def}{=} \sum_{\ell\in\mathbf{Z}} 2^{\frac{\ell}{2}}\left(\sum_{k=\ell-1}^{\infty}2^{-k}\|\Delta_k^h\Delta_\ell^v a\|^2_{L^4_h(L^2_v)}\right)^{\frac12}+\sum_{j\in\mathbf{Z}}2^{\frac{j}{2}}\|S_{j-1}^h\Delta_j^v a\|_{L^2(\mathbf{R}^3)}.$$

Definition 1.5 *We denote by* $\mathcal{B}_4^{-\frac12,\frac12}(T)$ *the space of* a *in* $C([0,T],\mathcal{B}_4^{-\frac12,\frac12})$ *such that*

$$
\begin{aligned}
\|a\|_{\mathcal{B}_4^{-\frac12,\frac12}(T)} \stackrel{\text{def}}{=} & \sum_{\ell\in\mathbf{Z}} 2^{\frac{\ell}{2}} \Biggl(\Bigl(\sum_{k=\ell-1}^{\infty} 2^{-k} \|\Delta_k^h \Delta_\ell^v a\|^2_{L^\infty_T(L^4_h(L^2_v))} \Bigr)^{\frac12} \\
& \qquad + \nu^{\frac12} \Bigl(\sum_{k=\ell-1}^{\infty} 2^{k} \|\Delta_k^h \Delta_\ell^v a\|^2_{L^2_T(L^4_h(L^2_v))} \Bigr)^{\frac12} \Biggr) \\
& + \sum_{j\in\mathbf{Z}} 2^{\frac{j}{2}} \Bigl(\|S^h_{j-1}\Delta^v_j a\|_{L^\infty_T(L^2(\mathbf{R}^3))} + \nu^{\frac12} \|\nabla_h S^h_{j-1}\Delta^v_j a\|_{L^2_T(L^2(\mathbf{R}^3))} \Bigr).
\end{aligned}
$$

In the following section, we shall use Littlewood-Paley theory to study the inner relations between $\mathcal{B}_4^{-\frac12,\frac12}(T)$ and $\mathcal{B}^{0,\frac12}(T)$. Now, we present the main results of this paper.

Theorem 1.2 *A constant* c *exists such that, if* $u_0 \in \mathcal{B}_4^{-\frac12,\frac12}$ *and* $\|u_0\|_{\mathcal{B}_4^{-\frac12,\frac12}} \leq c\nu$*, then, with initial data* u_0*, the system* (ANS_ν) *has a unique global solution* u *in* $\mathcal{B}_4^{-\frac12,\frac12}(\infty)$.

This theorem can be applied to initial data given by (1.2) thanks to the following proposition, proved in section 2.

Proposition 1.1 *Let* $\phi \in \mathcal{S}(\mathbf{R}^3)$*. If* $\phi_\varepsilon(x) \stackrel{\text{def}}{=} e^{ix_1/\varepsilon}\phi(x)$*, then* $\|\phi_\varepsilon\|_{\mathcal{B}_4^{-\frac12,\frac12}} = O(\varepsilon^{\frac12})$.

Classically, a global wellposedness theorem with small data in a space where smooth functions are dense corresponds to a version concerning local wellposedness for large data.

Theorem 1.3 *If* u_0 *belongs to* $\mathcal{B}_4^{-\frac12,\frac12}$*, then a positive* T *exists such that the system* (ANS_ν) *has a unique solution in the space* $\mathcal{B}_4^{-\frac12,\frac12}(T)$.

1.3 Structure of the text

This text does not contain all the details of the proofs and we refer to [7] for complete proofs. The purpose of the second section is to state some results about anisotropic Littlewood-Paley theory which will be of a constant use in what follows.

The third section will be devoted to the proof of the existence of a solution of (ANS_ν). In order to do it, we shall search for a solution of the form

$$
u = u_F + w \quad \text{with} \quad u_F \stackrel{\text{def}}{=} e^{\nu t\Delta_h} u_{hh}\,,\ u_{hh} \stackrel{\text{def}}{=} \sum_{k\geq \ell-1} \Delta_k^h\Delta_\ell^v u_0 \quad \text{and} \quad w\in \mathcal{B}^{0,\frac12}(\infty). \tag{1.4}
$$

In the last section, we shall prove the uniqueness in the following way. First, we shall establish a regularity theorem claiming that if $u \in \mathcal{B}_4^{-\frac12,\frac12}(T)$ is a solution of (ANS_ν) with initial data in $\mathcal{B}_4^{-\frac12,\frac12}$, then $w = u - u_F \in \mathcal{B}^{0,\frac12}(T)$. Therefore, looking at the equation of w, we shall prove the uniqueness of the solution u in the space $u_F + \mathcal{B}^{0,\frac12}(T)$. Let us point out that the uniqueness result we prove is surprinsingly not a stability result. We should mention that the method introduced by M. Paicu in [12] will play a crucial role in our proof here.

We present only the sketch of the proof and refer to [7] for the details.

Notations: Let A, B be two operators, we denote $[A; B] = AB - BA$, the commutator between A and B, $a \lesssim b$, we means that there is a uniform constant C, which may be different on different lines, such that $a \leq Cb$. Finally, we denote $L^r_T(L^p_h(L^q_v))$ the space $L^r([0,T]; L^p(\mathbf{R}_{x_1} \times \mathbf{R}_{x_2}; L^q(\mathbf{R}_{x_3})))$.

2 Some properties of anisotropic Littlewood-Paley theory

As we shall constantly use the anisotropic Littlewood-Paley theory, and in particular anisotropic Bernstein inequalities. We list them as the following:

Lemma 2.1 *Let $\mathcal{B}_h$ (resp. $\mathcal{B}_v$) a ball of $\mathbf{R}^2_h$ (resp. $\mathbf{R}_v$), and $\mathcal{C}_h$ (resp. $\mathcal{C}_v$) a ring of $\mathbf{R}^2_h$ (resp. $\mathbf{R}_v$); let $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q_2 \leq q_1 \leq \infty$. Then there holds:*
If the support of $\widehat{a}$ is included in $2^k\mathcal{B}_h$, then

$$\|\partial^\alpha_{x_h} a\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{k\left(|\alpha|+2\left(\frac{1}{p_2}-\frac{1}{p_1}\right)\right)} \|a\|_{L^{p_2}_h(L^{q_1}_v)}.$$

If the support of $\widehat{a}$ is included in $2^\ell\mathcal{B}_v$, then

$$\|\partial^\beta_3 a\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{\ell(\beta+(\frac{1}{q_2}-\frac{1}{q_1}))} \|a\|_{L^{p_1}_h(L^{q_2}_v)}.$$

If the support of $\widehat{a}$ is included in $2^k\mathcal{C}_h$, then

$$\|a\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{-kN} \sup_{|\alpha|=N} \|\partial^\alpha_h a\|_{L^{p_1}_h(L^{q_1}_v)}.$$

If the support of $\widehat{a}$ is included in $2^k\mathcal{C}_v$, then

$$\|a\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{-kN} \|\partial^N_3 a\|_{L^{p_1}_h(L^{q_1}_v)}.$$

Let us state two corollaries of this lemma, the proof of which are obvious and thus omitted.

Corollary 2.1 *The space $\mathcal{B}^{0,\frac12}$ is included in $\mathcal{B}_4^{-\frac12,\frac12}$ and so is $\mathcal{B}^{0,\frac12}(T)$ in $\mathcal{B}_4^{-\frac12,\frac12}(T)$ for any positive T. Moreover, the space $\mathcal{B}^{0,\frac12}(T)$ is included in $L^\infty_T(L^2_h(L^\infty_v))$.*

Corollary 2.2 *If a belongs to $\mathcal{B}_4^{-\frac12,\frac12}(T)$, then we have*

$$\sum_{\ell\in\mathbf{Z}} 2^{\frac{\ell}{2}} \Big(\sum_{k\in\mathbf{Z}} 2^{-k} \|\Delta^h_k\Delta^v_\ell a(0)\|^2_{L^4_h(L^2_v)}\Big)^{\frac12} \lesssim \|a(0)\|_{\mathcal{B}_4^{-\frac12,\frac12}} \quad \textit{and}$$

$$\sum_{\ell\in\mathbf{Z}} 2^{\frac{\ell}{2}} \bigg(\sum_{k\in\mathbf{Z}} \Big(2^{-k}\|\Delta^h_k\Delta^v_\ell a\|^2_{L^\infty_T(L^4_h(L^2_v))} + \nu 2^k \|\Delta^h_k\Delta^v_\ell a\|^2_{L^2_T(L^4_h(L^2_v))}\Big)\bigg)^{\frac12} \lesssim \|a\|_{\mathcal{B}_4^{-\frac12,\frac12}(T)}.$$

Proposition 1.1 tells us how large is the difference between the norms $\|\cdot\|_{\mathcal{B}^{0,\frac12}}$ and $\|\cdot\|_{\mathcal{B}_4^{-\frac12,\frac12}}$.

Proof of Proposition 1.1 By definition of the norm $\|\cdot\|_{\mathcal{B}_4^{-\frac12,\frac12}}$, we have, as the $\|\cdot\|_{\ell^2}$ norm is less than or equal to the $\|\cdot\|_{\ell^1}$ norm,

$$
\begin{aligned}
\|\phi_\varepsilon\|_{\mathcal{B}_4^{-\frac12,\frac12}} &\leq \sum_{j=1}^4 \Phi_\varepsilon^{(j)} \quad \text{with}\\
\Phi_\varepsilon^{(1)} &\stackrel{\text{def}}{=} \sum_{\substack{\varepsilon 2^k>1\\ k\geq \ell-1}} 2^{-\frac{k-\ell}{2}} \|\Delta_k^h\Delta_\ell^v\phi_\varepsilon\|_{L^4_h(L^2_v)},\\
\Phi_\varepsilon^{(2)} &\stackrel{\text{def}}{=} \sum_{\substack{\varepsilon 2^k\leq 1\\ k\geq \ell-1}} 2^{-\frac{k-\ell}{2}} \|\Delta_k^h\Delta_\ell^v\phi_\varepsilon\|_{L^4_h(L^2_v)},\\
\Phi_\varepsilon^{(3)} &\stackrel{\text{def}}{=} \sum_{\varepsilon 2^j>1} 2^{\frac j2}\|S_{j-1}^h\Delta_j^v\phi_\varepsilon\|_{L^2} \quad \text{and}\\
\Phi_\varepsilon^{(4)} &\stackrel{\text{def}}{=} \sum_{\varepsilon 2^j\leq 1} 2^{\frac j2}\|S_{j-1}^h\Delta_j^v\phi_\varepsilon\|_{L^2}.
\end{aligned}
$$

In order to estimate $\Phi_\varepsilon^{(1)}$, let us notice that

$$
\Phi_\varepsilon^{(1)} \leq \Big(\sum_{\varepsilon 2^k>1} 2^{-\frac k2}\Big)\sum_{\ell\in\mathbf{Z}} 2^{\frac\ell2}\sup_{k\in\mathbf{Z}}\|\Delta_k^h\Delta_\ell^v\phi_\varepsilon\|_{L^4_h(L^2_v)} \leq \varepsilon^{\frac12}\sum_{\ell\in\mathbf{Z}} 2^{\frac\ell2}\sup_{k\in\mathbf{Z}}\|\Delta_k^h\Delta_\ell^v\phi_\varepsilon\|_{L^4_h(L^2_v)}.
$$

Using Lemma 2.1, we have, by definition of ϕ_ε,

$$
\begin{aligned}
\sup_{k\in\mathbf{Z}}\|\Delta_k^h\Delta_\ell^v\phi_\varepsilon\|_{L^4_h(L^2_v)} &\lesssim \|\phi_\varepsilon\|_{L^4_h(L^2_v)} \lesssim \|\phi\|_{L^4_h(L^2_v)}. \quad \text{and}\\
\sup_{k\in\mathbf{Z}}\|\Delta_k^h\Delta_\ell^v\phi_\varepsilon\|_{L^4_h(L^2_v)} &\lesssim 2^{-\ell}\|\partial_3\phi_\varepsilon\|_{L^4_h(L^2_v)} \lesssim 2^{-\ell}\|\partial_3\phi\|_{L^4_h(L^2_v)}.
\end{aligned}
$$

Thus, taking the sum over $\ell\leq N$ and $\ell>N$ and choosing the best N gives

$$
\Phi_\varepsilon^{(1)} \leq \varepsilon^{\frac12}\sum_{\ell\in\mathbf{Z}} 2^{\frac\ell2}\sup_{k\in\mathbf{Z}}\|\Delta_k^h\Delta_\ell^v\phi_\varepsilon\|_{L^4_h(L^2_v)} \leq \varepsilon^{\frac12}\|\phi\|^{\frac12}_{L^4_h(L^2_v)}\|\partial_3\phi\|^{\frac12}_{L^4_h(L^2_v)}.
$$

The estimate of $\Phi_\varepsilon^{(2)}$ uses the oscillations. We have $\Delta_k^h\Delta_\ell^v\phi_\varepsilon = \phi_{k,\ell}^{1,\varepsilon}+\phi_{k,\ell}^{2,\varepsilon}$ with

$$
\begin{aligned}
\phi_{k,\ell}^{1,\varepsilon}(x) &\stackrel{\text{def}}{=} i\varepsilon\Delta_k^h\Delta_\ell^v(e^{i\frac{y_1}{\varepsilon}}\partial_1\phi) \quad \text{and}\\
\phi_{k,\ell}^{2,\varepsilon}(x) &\stackrel{\text{def}}{=} -i\varepsilon 2^k\widetilde{\phi}_{k,\ell}^{2,\varepsilon}(x) \quad \text{with}\\
\widetilde{\phi}_{k,\ell}^{2,\varepsilon}(x) &\stackrel{\text{def}}{=} 2^{2k}2^\ell\int(\partial_1\widetilde{g})(2^k(x_h-y_h))\widetilde{h}(2^\ell(x_3-y_3))e^{i\frac{y_1}{\varepsilon}}\phi(y)dy,
\end{aligned}
$$

where $(\widetilde{g},\widetilde{h})\in\mathcal{S}(\mathbf{R}^2)\times\mathcal{S}(\mathbf{R})$ such that $\mathcal{F}\widetilde{g}(\xi_h)=\widetilde{\varphi}(|\xi_h|)$ and $\mathcal{F}\widetilde{h}(\xi_3)=\widetilde{\varphi}(\xi_3)$. Using Lemma 2.1, we get

$$
2^{-\frac k2}\sum_{\ell\leq k+1} 2^{\frac\ell2}\|\phi_{k,\ell}^{1,\varepsilon}\|_{L^4_h(L^2_v)} \lesssim \varepsilon\sup_{\ell\in\mathbf{Z}}\|\Delta_k^h\Delta_\ell^v(e^{i\frac{y_1}{\varepsilon}}\partial_1\phi)\|_{L^4_h(L^2_v)} \lesssim \varepsilon 2^{\frac k2}\|\partial_1\phi\|_{L^2(\mathbf{R}^3)}.
$$

Moreover, we have

$$2^{-\frac{k}{2}} \sum_{\ell \le k+1} 2^{\frac{\ell}{2}} \|\phi^{2,\varepsilon}_{k,\ell}\|_{L^4_h(L^2_v)} \le \varepsilon 2^{\frac{k}{2}} \sum_{\ell \in \mathbf{Z}} 2^{\frac{\ell}{2}} \|\widetilde{\phi}^{2,\varepsilon}_{k,\ell}\|_{L^4_h(L^2_v)}.$$

Using Lemma 2.1, we get

$$\|\widetilde{\phi}^{2,\varepsilon}_{k,\ell}\|_{L^4_h(L^2_v)} \lesssim \|\phi\|_{L^4_h(L^2_v)} \quad \text{and} \quad \|\widetilde{\phi}^{2,\varepsilon}_{k,\ell}\|_{L^4_h(L^2_v)} \lesssim 2^{-\ell} \|\partial_3 \phi\|_{L^4_h(L^2_v)}.$$

Again taking the sum over $\ell \le N$ and $\ell > N$ and choosing the best N, we get

$$\sum_{\ell \in \mathbf{Z}} 2^{\frac{\ell}{2}} \|\widetilde{\phi}^{2,\varepsilon}_{k,\ell}\|_{L^4_h(L^2_v)} \lesssim \|\phi\|^{\frac{1}{2}}_{L^4_h(L^2_v)} \|\partial_3 \phi\|^{\frac{1}{2}}_{L^4_h(L^2_v)}.$$

We get that $\Phi^{(2)}_\varepsilon \le C_\phi \varepsilon \sum_{\varepsilon 2^k \le 1} 2^{\frac{k}{2}} \le C_\phi \varepsilon^{\frac{1}{2}}$. The estimates on $\Phi^{(3)}_\varepsilon$ and $\Phi^{(4)}_\varepsilon$ are analogous. ■

Notations In that follows, we make the convention that $(c_k)_{k \in \mathbf{Z}}$ (resp. $(d_j)_{j \in \mathbf{Z}}$) denotes a generic element of the sphere of $\ell^2(\mathbf{Z})$ (resp. $\ell^1(\mathbf{Z})$). Moreover, $(c_{k,\ell})_{(k,\ell) \in \mathbf{Z}^2}$ denotes a generic element of the sphere of $\ell^2(\mathbf{Z}^2)$ and $(d_{k,\ell})_{(k,\ell) \in \mathbf{Z}^2}$ denotes a generic sequence indexed by $\mathbf{Z}^2$ such that $\displaystyle\sum_{\ell \in \mathbf{Z}} \Big(\sum_{k \in \mathbf{Z}} d^2_{k,\ell} \Big)^{\frac{1}{2}} = 1$. Let us notice that we shall often use the following property, the easy proof of which is omitted.

Lemma 2.2 *Let α be a positive real number and N_0 an integer. Then we have*

$$\sum_{\substack{(k,\ell) \in \mathbf{Z}^2 \\ \ell \ge j - N_0}} 2^{-\alpha(\ell - j)} d_{k,\ell} c_k \lesssim d_j.$$

The following lemma will be of a frequent use in this work.

Lemma 2.3 *For any $a \in \mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)$, one has*

$$\|S^h_k \Delta^v_\ell a\|_{L^\infty_T(L^4_h(L^2_v))} + \nu^{\frac{1}{2}} \|\nabla_h S^h_k \Delta^v_\ell a\|_{L^2_T(L^4_h(L^2_v))} \lesssim d_{k,\ell} 2^{\frac{k}{2}} 2^{-\frac{\ell}{2}} \|a\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)} \quad \textit{and}$$
$$\|S^h_k a\|_{L^\infty_T(L^4_h(L^\infty_v))} + \nu^{\frac{1}{2}} \|\nabla_h S^h_k a\|_{L^2_T(L^4_h(L^\infty_v))} \lesssim c_k 2^{\frac{k}{2}} \|a\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)}.$$

With Lemma 2.3, we are going to state a result which is very close to Sobolev embedding and will be of a constant use in the existence proof of Theorem 1.2.

Lemma 2.4 *The space $\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)$ is included in $L^4_T(L^4_h(L^\infty_v))$. More precisely, let a be a function in $\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)$, then we have*

$$\|\Delta^v_j a\|_{L^4_T(L^4_h(L^2_v))} \lesssim \frac{d_j}{\nu^{\frac{1}{4}}} 2^{-\frac{j}{2}} \|a\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)} \quad \textit{and} \quad \|a\|_{L^4_T(L^4_h(L^\infty_v))} \lesssim \frac{1}{\nu^{\frac{1}{4}}} \|a\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)}.$$

Proof of Lemma 2.4 Let us first notice that $\|\Delta_j^v a\|^2_{L^4_T(L^4_h(L^2_v))} = \|(\Delta_j^v a)^2\|_{L^2_T(L^2_h(L^1_v))}$. Then using Bony's decomposition in the horizontal variables, we write

$$(\Delta_j^v a)^2 = \sum_{k\in\mathbf{Z}} S^h_{k-1}\Delta_j^v a \Delta_k^h \Delta_j^v a + \sum_{k\in\mathbf{Z}} S^h_{k+2}\Delta_j^v a \Delta_k^h \Delta_j^v a$$

These two terms are estimated exactly in the same way. Applying Hölder inequality, we get

$$\|S^h_{k-1}\Delta_j^v a \Delta_k^h \Delta_j^v a\|_{L^2_T(L^2_h(L^1_v))} \leq 2^{-\frac{k}{2}} \|S^h_{k-1}\Delta_j^v a\|_{L^\infty_T(L^4_h(L^2_v))} 2^{\frac{k}{2}} \|\Delta_k^h \Delta_j^v a\|_{L^2_T(L^4_h(L^2_v))}.$$

Using the first inequality of Lemma 2.3 and Corollary 2.2, we infer

$$\|S^h_{k-1}\Delta_j^v a \Delta_k^h \Delta_j^v a\|_{L^2_T(L^2_h(L^1_v))} \lesssim \frac{d^2_{k,j}}{\nu^{\frac{1}{2}}} 2^{-j} \|a\|^2_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)}.$$

Taking the sum over k and using Lemma 2.2, we deduce

$$\|(\Delta_j^v a)^2\|_{L^2_T(L^2_h(L^1_v))} \lesssim \frac{d^2_{j}}{\nu^{\frac{1}{2}}} 2^{-j} \|a\|^2_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)},$$

which is exactly the first inequality of the lemma. Now, using Lemma 2.1, we have

$$\|\Delta_j^v a\|_{L^4_T(L^4_h(L^\infty_v))} \lesssim 2^{\frac{j}{2}} \|\Delta_j^v a\|_{L^4_T(L^4_h(L^2_v))}.$$

This proves the whole lemma. ■

Now let us use Lemma 2.1 to study the free evolution u_F to the high horizontal frequency part of the initial data u_0, as defined in (1.4). In order to do so, let us first recall a lemma from [3] or [4], which describes the action of the semi-group of the heat equation on distributions, the Fourier transform of which are supported in a fixed ring.

Lemma 2.5 *Let $u_0 \in \mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}$ and u_F be as in (1.4), $\alpha \in \mathbf{N}^3, 1 \leq p \leq \infty$. Then, there holds*

$$\|\Delta_k^h \Delta_\ell^v u_F\|_{L^p_T(L^4_h(L^2_v))} \lesssim \begin{cases} \dfrac{d_{k,\ell}}{\nu^{\frac{1}{p}}} 2^{k\left(\frac{1}{2}-\frac{2}{p}\right)} 2^{-\frac{\ell}{2}} \|u_0\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}, & \text{for} \quad k \geq \ell - 1, \\ 0, \quad \text{otherwise.} \end{cases}$$

Moreover, u_F belongs to $\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(\infty)$, and we have $\|u_F\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(\infty)} \lesssim \|u_0\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}$.

Proof of Lemma 2.5 The relations (4) and (5) of the proof of Lemma 2.1 of [3] tell us that

$$\Delta_k^h \Delta_\ell^v u_F(t) = 2^{2k} g(t, 2^k\cdot) \star \Delta_k^h \Delta_\ell^v u_0 \quad \text{with} \quad \|g(t,\cdot)\|_{L^1(\mathbf{R}^2)} \leq Ce^{-c\nu t 2^{2k}}. \tag{2.1}$$

Here the convolution must be understood as the convolution on $\mathbf{R}^2$. Thus

$$\|\Delta_k^h \Delta_\ell^v u_F(t, x_h, \cdot)\|_{L^2_v} \leq 2^{2k} |g(t, 2^k\cdot)| \star \|\Delta_k^h \Delta_\ell^v u_0(\cdot)\|_{L^2_v}.$$

Using (2.1) and again Lemma 2.1, we get

$$\|\Delta_k^h \Delta_\ell^v u_F(t)\|_{L^4_h(L^2_v)} \lesssim e^{-c\nu t 2^{2k}} \|\Delta_k^h \Delta_\ell^v u_0\|_{L^4_h(L^2_v)} \lesssim e^{-c\nu t 2^{2k}} d_{k,\ell} 2^{\frac{k}{2}} 2^{-\frac{\ell}{2}} \|u_0\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}.$$

By integration, the lemma follows. ■

Lemma 2.6 *Under the assumptions of Lemma 2.5, one has*

$$\|\Delta_j^v u_F\|_{L^2(\mathbf{R}^+;L^\infty_h(L^2_v))} \lesssim \frac{d_j}{\sqrt{\nu}} 2^{-\frac{j}{2}} \|u_0\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} \quad \text{and} \quad \|u_F\|_{L^2(\mathbf{R}^+;L^\infty(\mathbf{R}^3))} \lesssim \frac{1}{\sqrt{\nu}} \|u_0\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}.$$

3 The proof of an existence theorem

The purpose of this section is to prove the following existence theorem.

Theorem 3.1 *A sufficiently small constant c exists which satisfies the following property: if u_0 is in $\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}$, and $\|u_0\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} \leq c\nu$, then the system (ANS_ν) has a global solution in the space $u_F + \mathcal{B}^{0,\frac{1}{2}}(\infty)$ where u_F is defined in (1.4).*

Proof of Theorem 3.1 As announced in the introduction, we shall look for a solution of the form $u = u_F + w$. Let us first establish the equation satisfied by w. Actually by substituting the above formula to (ANS_ν), we obtain

$$
(\widetilde{ANS_\nu}) \begin{cases} \partial_t w + w\cdot\nabla w - \nu\Delta_h w + w\cdot\nabla u_F + u_F\cdot\nabla w = -u_F\cdot\nabla u_F - \nabla p, \\ \operatorname{div} w = 0, \\ w|_{t=0} = u_{\ell h} \stackrel{\text{def}}{=} u_0 - u_{hh}. \end{cases}
$$

Notice that by (1.4), we have $u_{\ell h} = \sum_{j\in\mathbf{Z}} S^h_{j-1}\Delta^v_j u_0$. Moreover, there holds

$$
\Delta^v_j u_{\ell h} = \sum_{|j-j'|\leq 1} S^h_{j'-1}\Delta^v_{j'}\Delta^v_j u_0 \quad \text{and thus} \quad \|\Delta^v_j u_{\ell h}\|_{L^2} \lesssim \sum_{|j-j'|\leq 1} \|S^h_{j'-1}\Delta^v_{j'} u_0\|_{L^2}.
$$

This implies that, if u_0 belongs to $\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}$, then $u_{\ell h}$ belongs to $\mathcal{B}^{0,\frac{1}{2}}$ and

$$
\|u_{\ell h}\|_{\mathcal{B}^{0,\frac{1}{2}}} \lesssim \|u_0\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}. \tag{3.1}
$$

We shall use the classical Friedrichs' regularization method to construct the approximate solutions to $(\widetilde{ANS_\nu})$. For simplicity, we just outline it here (for the details in this context, see [12] or [4]). Let us define the sequence $(P_n)_{n\in\mathbf{N}}$ by $P_n a \stackrel{\text{def}}{=} \mathcal{F}^{-1}\big(\mathbf{1}_{B(0,n)}\widehat{a}\big)$ and

$$
(\widetilde{ANS_{\nu,n}}) \begin{cases} \partial_t w_n - \nu\Delta_h w_n + P_n(w_n\cdot\nabla w_n) + P_n(w_n\cdot\nabla u_{F,n}) + P_n(u_{F,n}\cdot\nabla w_n) \\ \quad = -P_n(u_{F,n}\cdot\nabla u_{F,n}) + P_n\nabla\Delta^{-1}\partial_j\partial_k\Big((u^j_{F,n}+w^j_n)(u^k_{F,n}+w^k_n)\Big) \\ \operatorname{div} w_n = 0, \\ w_n|_{t=0} = P_n(u_{\ell h}) \stackrel{\text{def}}{=} P_n(u_0 - u_{hh}). \end{cases}
$$

where $u_{F,n} \stackrel{\text{def}}{=} (\operatorname{Id} - S_{j_n})u_F$ with $j_n \sim -\log_2 n$ and where $\Delta^{-1}\partial_j\partial_k$ is defined precisely by

$$
\Delta^{-1}\partial_j\partial_k a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(|\xi|^{-2}\xi_j\xi_k\widehat{a}).
$$

Because of properties of L^2 and L^1 functions the Fourier transform of which are supported in the ball $B(0,n)$, the system $(\widetilde{ANS_{\nu,n}})$ appears to be an ordinary differential equation in

$$
L^2_n \stackrel{\text{def}}{=} \Big\{a\in L^2(\mathbf{R}^3)\,/\,\operatorname{Supp}\,\widehat{a}\subset B(0,n)\Big\}.
$$

This ordinary differential equation is globally wellposed because

$$
\frac{d}{dt}\|w_n(t)\|^2_{L^2} \leq C_n\|u_{F,n}(t)\|_{L^\infty}\|w_n\|^2_{L^2} + C_n\|u_{F,n}(t)\|^2_{L^4_h(L^2_v)}\|w_n(t)\|_{L^2}
$$

and $u_{F,n}$ belongs to $L^2(\mathbf{R}^+; L^\infty \cap L^4_h(L^2_v))$. We refer to [4] and [12] for the details. Now, the proof of Theorem 3.1 reduces to the following three propositions, which we admit for the time being.

Proposition 3.1 *Let u_0 be in $\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}$, and a in $\mathcal{B}^{0,\frac{1}{2}}(T)$. With u_F is defined in (1.4), we have*

$$\forall j \in \mathbf{Z}\,,\ I_j(T) \stackrel{\text{def}}{=} \int_0^T \big|\big(\Delta_j^v(u_F \cdot \nabla u_F)|\Delta_j^v a\big)\big|\, dt \lesssim \frac{d_j^2}{\nu} 2^{-j} \|u_0\|^2_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} \|a\|_{\mathcal{B}^{0,\frac{1}{2}}(T)}.$$

Proposition 3.2 *Let a be a divergence free vector in $\mathcal{B}^{0,\frac{1}{2}}(T)$, and b in $\mathcal{B}^{0,\frac{1}{2}}(T)$. Then*

$$\forall j \in \mathbf{Z}\,,\ J_j(T) \stackrel{\text{def}}{=} \int_0^T \big|\big(\Delta_j^v(a \cdot \nabla u_F)|\Delta_j^v b\big)\big|\, dt \lesssim \frac{d_j^2}{\nu} 2^{-j} \|a\|_{\mathcal{B}^{0,\frac{1}{2}}(T)} \|u_0\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} \|b\|_{\mathcal{B}^{0,\frac{1}{2}}(T)}.$$

Proposition 3.3 *Let a be a divergence free vector in $\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)$, and $b \in \mathcal{B}^{0,\frac{1}{2}}(T)$. Then*

$$\forall j \in \mathbf{Z}\,,\ F_j(T) \stackrel{\text{def}}{=} \int_0^T \big|\big(\Delta_j^v(a \cdot \nabla b)|\Delta_j^v b\big)\big|\, dt \lesssim \frac{d_j^2}{\nu} 2^{-j} \|a\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)} \|b\|^2_{\mathcal{B}^{0,\frac{1}{2}}(T)}.$$

Conclusion of the proof of Theorem 3.1 Notice from $(\widetilde{ANS}_{\nu,n})$ that $P_n w_n = w_n$, we apply the operator Δ_j^v to $(\widetilde{ANS}_{\nu,n})$ and take the L^2 inner product of the resulting equation with $\Delta_j^v w_n$ to get

$$\begin{aligned}&\frac{d}{dt}\|\Delta_j^v w_n(t)\|^2_{L^2} + 2\nu\|\nabla_h \Delta_j^v w_n(t)\|^2_{L^2} = -2(\Delta_j^v(w_n \cdot \nabla w_n)|\Delta_j^v w_n)\\ &- 2(\Delta_j^v(u_{F,n} \cdot \nabla w_n)|\Delta_j^v w_n) - 2(\Delta_j^v(w_n \cdot \nabla u_{F,n})|\Delta_j^v w_n) - 2(\Delta_j^v(u_{F,n} \cdot \nabla u_{F,n})|\Delta_j^v w_n).\end{aligned}$$

By integration the above equation over $[0,T]$, we get

$$2^j\|\Delta_j^v w_n\|^2_{L^\infty_T(L^2)} + 2^{j+1}\nu\|\nabla_h \Delta_j^v w_n\|^2_{L^2_T(L^2)} \leq 2^j\|\Delta_j^v w_n(0)\|^2_{L^2} + 2\sum_{k=1}^4 W_j^k(T) \tag{3.2}$$

with

$$\begin{aligned}
W_j^1(T) &\stackrel{\text{def}}{=} 2^j \int_0^T \Big|(\Delta_j^v(w_n(t) \cdot \nabla w_n(t))|\Delta_j^v w_n(t))\Big| dt,\\
W_j^2(T) &\stackrel{\text{def}}{=} 2^j \int_0^T \Big|(\Delta_j^v(u_{F,n}(t) \cdot \nabla w_n(t))|\Delta_j^v w_n(t))\Big| dt,\\
W_j^3(T) &\stackrel{\text{def}}{=} 2^j \int_0^T \Big|(\Delta_j^v(w_n(t) \cdot \nabla u_{F,n}(t))|\Delta_j^v w_n(t))\Big| dt,\\
W_j^4(T) &\stackrel{\text{def}}{=} 2^j \int_0^T \Big|(\Delta_j^v(u_{F,n}(t) \cdot \nabla u_{F,n}(t))|\Delta_j^v w_n(t))\Big| dt.
\end{aligned}$$

Proposition 3.3 applied with $a = b = w_n$ together with Corollary 2.1 gives

$$W_j^1(T) \lesssim \frac{d_j^2}{\nu}\|w_n\|^3_{\mathcal{B}^{0,\frac{1}{2}}(T)}. \tag{3.3}$$

Thanks to Lemma 2.5, Proposition 3.3 applied with $a = u_{F,n}$ and $b = w_n$, then Proposition 3.2 applied with $a = b = w_n$ and finally Proposition 3.1implies in particular that

$$
\begin{aligned}
W_j^2(T) &\lesssim d_j^2 \frac{\|u_0\|_{\mathcal{B}_4^{-\frac12,\frac12}}}{\nu} \|w_n\|^2_{\mathcal{B}^{0,\frac12}(T)},\\
W_j^3(T) &\lesssim d_j^2 \frac{\|u_0\|_{\mathcal{B}_4^{-\frac12,\frac12}}}{\nu} \|w_n\|^2_{\mathcal{B}^{0,\frac12}(T)} \quad \text{and}\\
W_j^4(T) &\lesssim d_j^2 \frac{\|u_0\|^2_{\mathcal{B}_4^{-\frac12,\frac12}}}{\nu} \|w_n\|_{\mathcal{B}^{0,\frac12}(T)}.
\end{aligned}
\tag{3.4}
$$

Plugging estimates (3.3) and (3.4) into (3.2) gives

$$
\begin{aligned}
2^j &\Big(\|\Delta_j^v w_n\|_{L^\infty_T(L^2)} + \sqrt{2\nu}\|\nabla_h \Delta_j^v w_n\|_{L^2_T(L^2)}\Big)^2\\
&\leq 2^j\|\Delta_j^v w_n(0)\|^2_{L^2} + \frac{C}{\nu} d_j^2 \left(\|w_n\|^2_{\mathcal{B}^{0,\frac12}(T)} + \|u_0\|^2_{\mathcal{B}_4^{-\frac12,\frac12}}\right)\|w_n\|_{\mathcal{B}^{0,\frac12}(T)}.
\end{aligned}
$$

Using (3.1), we get, by definition of $\mathcal{B}^{0,\frac12}(T)$,

$$
\|w_n\|_{\mathcal{B}^{0,\frac12}(T)} \leq 2C_0\|u_0\|_{\mathcal{B}_4^{-\frac12,\frac12}} + \frac{C}{\sqrt{\nu}}\left(\|w_n\|_{\mathcal{B}^{0,\frac12}(T)} + \|u_0\|_{\mathcal{B}_4^{-\frac12,\frac12}}\right)\|w_n\|^{\frac12}_{\mathcal{B}^{0,\frac12}(T)}. \tag{3.5}
$$

Let us define

$$
T_n \stackrel{\text{def}}{=} \sup\left\{T > 0 \,/\, \|w_n\|_{\mathcal{B}^{0,\frac12}(T)} \leq 4C_0\|u_0\|_{\mathcal{B}_4^{-\frac12,\frac12}}\right\}.
$$

The fact that w_n is continuous with value in H^N for any integer N implies that T_n is positive. Then, Inequality (3.5) implies that, for any n and for any $T < T_n$, we have

$$
\|w_n\|_{\mathcal{B}^{0,\frac12}(T)} \leq 2C_0\|u_0\|_{\mathcal{B}_4^{-\frac12,\frac12}} + \frac{2C(4C_0+1)\sqrt{C_0}}{\sqrt{\nu}}\|u_0\|^{\frac32}_{\mathcal{B}_4^{-\frac12,\frac12}}.
$$

Then, if $\|u_0\|_{\mathcal{B}_4^{-\frac12,\frac12}}$ is small enough with respect to ν, we get, for any n and for any $T < T_n$,

$$
\|w_n\|_{\mathcal{B}^{0,\frac12}(T)} \leq \frac52 C_0\|u_0\|_{\mathcal{B}_4^{-\frac12,\frac12}}.
$$

Thus $T_n = +\infty$ for any n. Then, the existence follows from classical compactness methods, the details of which are omitted (see [12] or [4]). Then, Theorem 3.1 is proved, provided of course that we have proved the three propositions 3.1–3.3. ∎

In the proof of the above three propooitions, things are different for terms involving horizontal derivative and for terms involving vertical derivatives. Let us only prove Propositon 3.1 just to give a idea of the methods. It relies on the following lemma.

Lemma 3.1 *Let (a,b) be in $\Big(\mathcal{B}_4^{-\frac12,\frac12}(T)\Big)^2$. We have*

$$
\|\Delta_j^v(ab)\|_{L^2_T(L^2)} \lesssim \frac{d_j}{\nu^{\frac12}} 2^{-\frac{j}{2}}\|a\|_{\mathcal{B}_4^{-\frac12,\frac12}(T)}\|b\|_{\mathcal{B}_4^{-\frac12,\frac12}(T)}.
$$

Proof of Lemma 3.1 Let us write

$$\Delta_j^v(ab) = \sum_{|j'\geq j-N_0} \Delta_j^v\left(S_{j'}^v a\Delta_{j'}^v b) + \Delta_j^v(S_{j'+1}^v b\Delta_{j'}^v a\right).$$

Using Hölder inequality and Lemma 2.4, we get

$$\begin{aligned}2^{\frac{j}{2}}\|\Delta_j^v \sum_{j'\geq j-N_0}(S_{j'}^v a\Delta_{j'}^v b)\|_{L_T^2(L_h^2(L_v^2))} &\lesssim \|S_{j'}^v a\|_{L_T^4(L_h^4(L_v^\infty))}\|\Delta_{j'}^v b\|_{L_T^4(L_h^4(L_v^2))}\\ &\lesssim \Big(\sum_{j'\geq j-N_1} 2^{-\frac{j'-j}{2}} d_{j'}\Big)\nu^{\frac{1}{2}}2^{-\frac{j'}{2}}\|a\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)}\|b\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)}.\end{aligned}$$

The lemma is proved. ■

Proof of Proposition 3.1 Thanks to the fact that u_F is divergence free, we have,

$$I_j(T) = \int_0^T \left|\left(\Delta_j^v(u_F\cdot\nabla u_F)|\Delta_j^v a\right)\right| dt \leq I_j^h(T) + I_j^v(T), \quad \text{with}$$

$$I_j^h(T) \stackrel{\text{def}}{=} \int_0^T \left|\Big(\Delta_j^v(u_F^h\otimes u_F)|\Delta_j^v\nabla_h a\Big)\right| dt \quad\text{and}\quad I_j^v(T) \stackrel{\text{def}}{=} \int_0^T \left|\left(\partial_3\Delta_j^v(u_F^3 u_F)|\Delta_j^v a\right)\right| dt.$$

Using Lemmas 2.5 and 3.1, we get

$$I_j^h(T) \leq \|\Delta_j^v(u_F^h u_F)\|_{L_T^2(L^2)}\|\Delta_j^v(\nabla_h a)\|_{L_T^2(L^2)} \lesssim \frac{d_j^2}{\nu}2^{-j}\|u_0\|^2_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}\|a\|_{\mathcal{B}^{0,\frac{1}{2}}(T)}.$$

For the term with the vertical derivative, let us write, using Lemma 2.1,

$$I_j^v(T) \lesssim 2^j\|\Delta_j^v(u_F^3 u_F)\|_{L_T^1(L^2)}\|\Delta_j^v b\|_{L_T^\infty(L^2)}.$$

Using again Bony's decomposition, we infer

$$\begin{aligned}\Delta_j^v(u_F^3 u_F) &= \sum_{|j'-j|\leq 5}\Delta_j^v(S_{j'-1}^v u_F^3\Delta_{j'}^v u_F) + \sum_{j'\geq j-N_0}\Delta_j^v(\Delta_{j'}^v u_F^3 S_{j'+2}^v u_F) \quad\text{and}\\ S_{j'-1}^v u_F\Delta_{j'}^v u_F &= \sum_{k\geq j'-N_0}\Big\{S_{k-1}^h S_{j'-1}^v u_F^3\Delta_k^h\Delta_{j'}^v u_F + \Delta_k^h S_{j'-1}^v u_F^3 S_{k+2}^h\Delta_{j'}^v u_F\Big\}.\end{aligned}$$

The two terms of the above sum are estimated exacty along the same lines. As in the proof of Lemma 2.6, we use the smoothing effect on u_F on the highest possible horizontal frequencies. Using Hölder inequality, this gives

$$\|S_{k-1}^h S_{j'-1}^v u_F^3\Delta_k^h\Delta_{j'}^v u_F\|_{L_T^1(L^2)} \leq 2^{-\frac{k}{2}}\|S_{k-1}^h S_{j'-1}^v u_F\|_{L_T^\infty(L_h^4(L_v^\infty))}2^{\frac{k}{2}}\|\Delta_k^h\Delta_{j'}^v u_F\|_{L_T^1(L_h^4(L_v^2))}.$$

Lemma 2.5 and Lemma 2.3 give

$$\begin{aligned}2^{\frac{k}{2}}\|\Delta_k^h\Delta_{j'}^v u_F\|_{L_T^1(L_h^4(L_v^2))} &\lesssim \frac{1}{\nu}d_{k,j}2^{-\frac{j'}{2}}2^{-k}\|u_0\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} \quad\text{and}\\ 2^{-\frac{k}{2}}\|S_{k-1}^h S_{j'-1}^v u_F\|_{L_T^\infty(L_h^4(L_v^\infty))} &\lesssim c_k\|u_0\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}.\end{aligned}$$

Then, using, it turns out that

$$\begin{aligned}\|S^v_{j'-1}u_F\Delta^v_{j'}u_F\|_{L^1_T(L^2)} &\lesssim \frac{1}{\nu}\Big(\sum_{k\geq j'-1} c_k d_{k,j'}2^{-k}\Big)2^{-\frac{j'}{2}}\|u_0\|^2_{\mathcal{B}_4^{-\frac12,\frac12}}\\ &\lesssim \frac{d_{j'}}{\nu}2^{-\frac{3j'}{2}}\|u_0\|^2_{\mathcal{B}_4^{-\frac12,\frac12}}.\end{aligned}$$

We deduce that

$$2^{\frac{3j}{2}}\|\Delta^v_j(u_F^3u_F)\|_{L^1_T(L^2)} \lesssim \frac{1}{\nu}\|u_0\|^2_{\mathcal{B}_4^{-\frac12,\frac12}}\sum_{j'\geq j-N_1} d_{j'}2^{-\frac{3(j'-j)}{2}}.$$

This concludes the proof of Proposition 3.1. ■

4 The proof of the uniqueness

The first step in order to prove the uniqueness part of Theorems 1.2 and 1.3 is the proof of the following regularity theorem.

Theorem 4.1 *Let $u \in \mathcal{B}_4^{-\frac12,\frac12}(T)$ be a solution of (ANS_ν) with initial data u_0 in $\mathcal{B}_4^{-\frac12,\frac12}$. Then, if u_F is defined by (1.4), we have $w = u - u_F \in \mathcal{B}^{0,\frac12}(T)$.*

The above theorem implies that, if u_j are two solutions of (ANS_ν) in $\mathcal{B}_4^{-\frac12,\frac12}(T)$ associated with the same initial data, then $\delta \stackrel{\text{def}}{=} u_2 - u_1$ belongs to $\mathcal{B}^{0,\frac12}(T)$. Moreover, it satisfies the following system

$$(ANS'_\nu)\begin{cases}\partial_t\delta - \nu\Delta_h\delta = L\delta - \nabla p\\ \operatorname{div}\delta = 0\\ \delta_{|t=0} = 0\end{cases}$$

where L is the following linear operator $L\delta \stackrel{\text{def}}{=} -\delta\nabla u_1 - u_2\nabla\delta$. In order to prove uniqueness, we have to prove that $\delta \equiv 0$. Because the existence of solution to (ANS_ν) is not proved by using Picard's fixed point method, the uniqueness can not be given by a contraction in the space $\mathcal{B}^{0,\frac12}$ or even $\mathcal{B}_4^{-\frac12,\frac12}$. As pointed out first by D. Iftimie in [10], the system (ANS_ν) is hyperbolic in the vertical direction. Roughly speaking, for hyperbolic system, the contraction argument can be realized with one less derivative than the existence space. Here of course, the derivative is lost in the vertical direction. The first idea is the introduction of the homogenenous norm, given in the following definition.

Definition 4.1 *Let $s \in \mathbf{R}$, let us define the following semi norm*

$$\|a\|_{H^{0,s}} \stackrel{\text{def}}{=} \Big(\sum_{j\in\mathbf{Z}} 2^{2js}\|\Delta^v_j a\|^2_{L^2}\Big)^{\frac12}.$$

Remark It is obvious that

$$\|a\|^2_{L^\infty_T(H^{0,\frac12})} + \nu\|\nabla_h a\|^2_{L^2_T(H^{0,\frac12})} \lesssim \|a\|^2_{\mathcal{B}^{0,\frac12}(T)}. \tag{4.1}$$

The norm $\|\cdot\|_{H^{0,-\frac{1}{2}}}$ is not very convenient to work with. In particular, it carries on informations about low frequencies which is not necesseraly relevant in the proof of an uniqueness theorem which is by definition a local result. Moreover, there is no evidence that δ belongs to such a space. We bypass this problem by the introduction of the inhomogenenous version of the above norm. In order to do it, let us introduce the following notations:

$$\Delta_j^{vi} = \Delta_j^v\,,\ S_j^{vi} = S_j^v \quad \text{if} \quad j \geq 0 \quad \text{and} \quad \Delta_j^{vi} = S_{j+1}^{vi} = 0 \quad \text{if} \quad j \leq -2.$$

This leads to the following definition of the norm, which we use for a contraction argument.

Definition 4.2 *Let us denote by $\mathcal{H}$ the space of tempered distribution such that*

$$\|a\|_{\mathcal{H}}^2 \stackrel{\text{def}}{=} \sum_{j\in\mathbf{Z}} 2^{-j}\|\Delta_j^{vi}a\|_{L^2}^2 < \infty.$$

Now the key point is the estimate of $(L\delta|\delta)_{\mathcal{H}}$. We follow mainly [12] up to the fact that the solutions u_1 and u_2 are not in $\mathcal{B}^{0,\frac{1}{2}}(T)$ but only in $\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)$. This leads to the following definition.

Definition 4.3 *Let us denote by $\mathcal{B}_u$ the following (semi) norm*

$$\|b\|_{\mathcal{B}_u}^2 \stackrel{\text{def}}{=} \sum_{\substack{k\in\mathbf{Z}\\ j\in\mathbf{N}}} 2^{j-k}\|\Delta_k^h\Delta_j^v a\|_{L_h^4(L_v^2)}^2.$$

Remark We obviously have

$$\|b\|_{L_T^\infty(\mathcal{B}_u)}^2 + \nu\|\nabla_h b\|_{L_T^2(\mathcal{B}_u)}^2 \lesssim \|b\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)}^2. \tag{4.2}$$

Let us state and admit (see [7] for the details) the following variation of Lemma 3.2 of [12].

Lemma 4.1 *Let a and b be divergence free vector fields such that a and $\nabla_h a$ are in $H^{0,\frac{1}{2}}\cap\mathcal{H}$, b is in $\mathcal{B}_u \cap L_h^4(L_v^\infty)$ with $\nabla_h b \in \mathcal{B}_u$. Let us assume also that $\|a\|_{\mathcal{H}}^2 \leq 2^{-16}$. Then we have*

$$|(b\cdot\nabla a|a)_{\mathcal{H}}| + |(a\cdot\nabla b|a)_{\mathcal{H}}| \leq \frac{\nu}{10}\|\nabla_h a\|_{\mathcal{H}}^2 + C(a,b)\mu(\|a\|_{\mathcal{H}}^2)$$

with $\mu(r) \stackrel{\text{def}}{=} r(1-\log_2 r)\log_2(1-\log_2 r)$ and

$$\begin{aligned} C(a,b) \stackrel{\text{def}}{=} {} & \frac{C}{\nu}\|b\|_{L_h^4(L_v^\infty)}^2\Big(1+\frac{\|b\|_{L_h^4(L_v^\infty)}^2}{\nu^2}\Big) \\ & + \frac{C}{\nu}(1+\|b\|_{\mathcal{B}_u}^2)\Big(1+\frac{\|b\|_{\mathcal{B}_u}^4}{\nu^2}\Big)\Big(\|b\|_{\mathcal{B}_u}^2\|\nabla_h b\|_{\mathcal{B}_u}^2 + \|a\|_{H^{0,\frac{1}{2}}}^2\|\nabla_h a\|_{H^{0,\frac{1}{2}}}^2\Big). \end{aligned}$$

Conclusion of the proof of Theorem 1.3 We postpone the proof of the fact that

$$\delta \in L_T^\infty(\mathcal{H}) \quad \text{and} \quad \nabla_h\delta \in L_T^2(\mathcal{H}) \tag{4.3}$$

which is a low vertical frequency information on δ. Lemma 4.1 implies that, for any $t\in[0,T]$,

$$\|\delta(t)\|_{\mathcal{H}}^2 \leq \int_0^t f(t')\mu(\|\delta(t')\|_{\mathcal{H}}^2)dt' \quad \text{with} \quad f(t) \stackrel{\text{def}}{=} C(u_1(t),\delta(t)) + C(u_2(t),\delta(t)).$$

Lemma 2.4 and assertions (4.1) and (4.2) imply that $f \in L^1([0,T])$. Thus Theorems 1.2 and 1.3 are proved, provided of course that we prove Assertion (4.3). ■

Proof of Assertion (4.3) Let us write that $S_0^v\delta$ is a solution (with initial value 0) of

$$\begin{aligned}
\partial_t S_0^v\delta - \nu\Delta_h S_0^v\delta &= g_1 + g_2 + g_3 \quad \text{with}\\
g_1 &\stackrel{\text{def}}{=} \sum_{\lambda\in\Lambda} S_0^v\partial_3(a_\lambda b_\lambda)\\
g_2 &\stackrel{\text{def}}{=} \sum_{\lambda\in\Lambda} S_0^v\partial_h(c_\lambda(\mathrm{Id}-S_0^v)\delta) \quad \text{and}\\
g_3 &\stackrel{\text{def}}{=} S_0^v\partial_h\sum_{\lambda\in\Lambda} d_\lambda S_0^v\delta
\end{aligned}$$

where Λ is a finite set of indices and a_λ, b_λ, c_λ and d_λ belong to $\mathcal{B}_4^{-\frac12,\frac12}(T)$. Using Lemmas 2.1 and 3.1, we get that

$$\begin{aligned}
\|S_0^v\partial_3(a_\lambda b_\lambda)\|_{L^2_T(L^2)} &\lesssim \sum_{j\leq -1} 2^j\|\Delta_j^v(a_\lambda b_\lambda)\|_{L^2_T(L^2)}\\
&\lesssim \frac{1}{\nu^{\frac12}}\|a_\lambda\|_{\mathcal{B}_4^{-\frac12,\frac12}(T)}\|b_\lambda\|_{\mathcal{B}_4^{-\frac12,\frac12}(T)}.
\end{aligned}$$

Thus we have that

$$\|g_1\|_{L^2_T(L^2)} \lesssim \frac{1}{\nu^{\frac12}}C_{12}^2(T) \quad \text{with} \quad C_{12}(T) \stackrel{\text{def}}{=} \|u_1\|_{\mathcal{B}_4^{-\frac12,\frac12}(T)} + \|u_2\|_{\mathcal{B}_4^{-\frac12,\frac12}(T)}. \tag{4.4}$$

We estimate g_2 using Lemma 2.4. It claims in particular that

$$\|(\mathrm{Id}-S_0^v)\delta\|_{L^4_T(L^4_h(L^2_v))} \lesssim \frac{1}{\nu^{\frac14}}\Big(\sum_{j\geq 0} 2^{-\frac{j}{2}}\Big)\|\delta\|_{\mathcal{B}_4^{-\frac12,\frac12}(T)} \lesssim \frac{1}{\nu^{\frac14}}\|\delta\|_{\mathcal{B}_4^{-\frac12,\frac12}(T)}.$$

Lemma 2.4 also claims that $\|c_\lambda\|_{L^4_T(L^4_h(L^\infty_v))} \lesssim \frac{1}{\nu^{\frac14}}\|c_\lambda\|_{\mathcal{B}_4^{-\frac12,\frac12}(T)}$. Then we have

$$\begin{aligned}
\|c_\lambda(\mathrm{Id}-S_0^v)\delta\|_{L^2_T(L^2)} &\lesssim \|c_\lambda\|_{L^4_T(L^4_h(L^\infty_v))}\|(\mathrm{Id}-S_0^v)\delta\|_{L^4_T(L^4_h(L^2_v))}\\
&\lesssim \frac{1}{\nu^{\frac12}}\|c_\lambda\|_{\mathcal{B}_4^{-\frac12,\frac12}(T)}\|\delta\|_{\mathcal{B}_4^{-\frac12,\frac12}(T)}.
\end{aligned}$$

This gives that

$$g_2 = \mathrm{div}_h\,\widetilde{g}_2 \quad \text{with} \quad \|\widetilde{g}_2\|_{L^2_T(L^2)} \lesssim \frac{1}{\nu^{\frac12}}C_{12}^2(T). \tag{4.5}$$

The term g_3 must be treated with a commutator argument based on the following lemma.

Lemma 4.2 *Let χ be a function of $\mathcal{S}(\mathbf{R})$. A constant C exists such that, for any function a in $L^2_h(L^\infty_v)$, we have*

$$\|[\chi(\varepsilon x_3); S_0^v]a\|_{L^2} \leq C\varepsilon^{\frac12}\|a\|_{L^2_h(L^\infty_v)}.$$

Proof of Lemma 4.2 Taylor's formula at order one gives

$$
\begin{aligned}
C_\varepsilon(a)(x_h,x_3) &\stackrel{\text{def}}{=} [\chi(\varepsilon x_3); S_0^v]a(x_h,x_3)\\
&= \varepsilon\int_{\mathbf{R}\times[0,1]} h(x_3-y_3)\chi'\left(\varepsilon((1-\tau)x_3+\tau y_3)\right)a(x_h,y_3)dy_3d\tau.
\end{aligned}
$$

Cauchy-Schwarz inequality for the measure $|h(x_3-y_3)|dx_3dy_3d\tau$ on $\mathbf{R}^2\times[0,1]$ gives

$$
\begin{aligned}
\|C_\varepsilon(a)(x_h,\cdot)\|_{L^2_v}^2 &\leq \varepsilon^2\|a(x_h,\cdot)\|_{L^\infty_v}^2 \sup_{\|\varphi\|_{L^2(\mathbf{R})}\leq 1}\left(\int_{\mathbf{R}^2}|h(x_3-y_3)|\varphi^2(x_3)dx_3dy_3\right)(H_1^\varepsilon+H_2^\varepsilon)\\
&\leq C\varepsilon^2\|a(x_h,\cdot)\|_{L^\infty_v}^2(H_1^\varepsilon+H_2^\varepsilon)\quad\text{with}\\
H_1^\varepsilon &\stackrel{\text{def}}{=} \int_{\mathbf{R}^2\times[0,\frac12]}(\chi')^2\left(\varepsilon((1-\tau)x_3+\tau y_3)\right)|h(x_3-y_3)|dx_3dy_3d\tau\quad\text{and}\\
H_2^\varepsilon &\stackrel{\text{def}}{=} \int_{\mathbf{R}^2\times[\frac12,1]}(\chi')^2\left(\varepsilon((1-\tau)x_3+\tau y_3)\right)|h(x_3-y_3)|dx_3dy_3d\tau.
\end{aligned}
$$

Changing variables

$$
\left\{\begin{array}{lcl} x_\tau &=& (1-\tau)x_3+\tau y_3\\ y_\tau &=& y_3\end{array}\right.\quad\text{in } H_1^\varepsilon\quad\text{and}\quad\left\{\begin{array}{lcl} x_\tau &=& x_3\\ y_\tau &=& \tau y_3+(1-\tau)x_3\end{array}\right.\quad\text{in } H_2^\varepsilon
$$

gives

$$
\begin{aligned}
H_1^\varepsilon &= \int_{\mathbf{R}^2\times[0,\frac12]}\frac{1}{1-\tau}(\chi')^2(\varepsilon x_\tau)\Big|h\Big(\frac{x_\tau-y_\tau}{1-\tau}\Big)\Big|dx_\tau dy_\tau d\tau\quad\text{and}\\
H_2^\varepsilon &= \int_{\mathbf{R}^2\times[\frac12,1]}\frac{1}{\tau}(\chi')^2(\varepsilon y_\tau)\Big|h\Big(\frac{x_\tau-y_\tau}{\tau}\Big)\Big|dx_\tau dy_\tau d\tau.
\end{aligned}
$$

We immediately infer that $\|C_\varepsilon(a)(x_h,\cdot)\|_{L^2_v}\leq C\varepsilon^{\frac12}\|a(x_h,\cdot)\|_{L^\infty_v}$ and the lemma is proved. ■

Now let us choose $\chi\in\mathcal{D}(\mathbf{R})$ with value 1 near 0 and let us state $S_{0,\varepsilon}^v a\stackrel{\text{def}}{=}\chi(\varepsilon\cdot)S_0^v a$. The classical L^2 energy estimate gives

$$
\begin{aligned}
\|S_{0,\varepsilon}^v\delta(t)\|_{L^2}^2+\nu\int_0^t\|\nabla_h S_{0,\varepsilon}^v\delta(t')\|_{L^2}^2dt' &\leq 2\int_0^t\|g_1(t')\|_{L^2}\|S_{0,\varepsilon}^v\delta(t')\|_{L^2}dt'\\
&\quad+\frac1\nu\int_0^t\|\widetilde{g}_2(t')\|_{L^2}^2dt'+2\int_0^t\langle\chi(\varepsilon\cdot)g_3(t'),S_{0,\varepsilon}^v\delta(t')\rangle dt'.
\end{aligned}
$$

By definition of g_3, the integrand in the last term of the above equality is a finite sum of terms of the type

$$
D_\lambda\stackrel{\text{def}}{=}\langle\chi(\varepsilon\cdot)S_0^v(d_\lambda S_0^v\delta),\partial_h S_{0,\varepsilon}^v\delta\rangle
$$

with $d_\lambda\in\mathcal{B}_4^{-\frac12,\frac12}(T)$. Writing that $D_\lambda=D_\lambda^1+D_\lambda^2$ with

$$
D_\lambda^1\stackrel{\text{def}}{=}\langle[\chi(\varepsilon\cdot);S_0^v](d_\lambda S_0^v)\delta,\partial_h S_{0,\varepsilon}^v\delta\rangle\quad\text{and}\quad D_\lambda^2\stackrel{\text{def}}{=}\langle S_0^v(d_\lambda S_{0,\varepsilon}^v\delta),\partial_h S_{0,\varepsilon}^v\delta\rangle.
$$

Lemmas 2.4 and 4.2 imply that

$$\begin{aligned}\int_0^t |D^1_\lambda(t')|dt' &\lesssim \varepsilon^{\frac12} C^2_{12}(t)\|\nabla_h S_{0,\varepsilon}\delta\|_{L^2_t(L^2)}\\ &\leq \frac{\nu}{4}\|\nabla_h S^v_{0,\varepsilon}\delta\|_{L^2_t(L^2)} + \frac{C}{\nu}\varepsilon C^4_{12}(t).\end{aligned}$$

Then let us write that

$$\begin{aligned}|D^2_\lambda(t)| &\lesssim \|d_\lambda(t)\|_{L^4_h(L^2_v)}\|S^v_{0,\varepsilon}\delta(t)\|^{\frac12}_{L^2}\|\nabla_h S^v_{0,\varepsilon}\delta(t)\|^{\frac32}_{L^2}\\ &\leq \frac{\nu}{4}\|\nabla_h S^v_{0,\varepsilon}(t)\|^2_{L^2} + \frac{C}{\nu^3}\|d_\lambda(t)\|^4_{L^4_h(L^\infty_v)}\|S^v_{0,\varepsilon}(t)\|^2_{L^2}.\end{aligned}$$

Using (4.4) we get

$$\begin{aligned}\|S^v_{0,\varepsilon}\delta(t)\|^2_{L^2} + \frac{\nu}{2}\int_0^t \|\nabla_h S^v_{0,\varepsilon}\delta(t')\|^2_{L^2}dt' &\leq \frac{C}{\nu}(\varepsilon+1)C^4_{12}(T) + C\int_0^t \|g_1(t')\|^2_{L^2}\,dt'\\ &+ C\int_0^t\Big(1+\frac{1}{\nu^3}\left(\|u_1\|^4_{L^4_h(L^\infty_v)} + \|u_2\|^4_{L^4_h(L^\infty_v)}\right)\Big)\|S^v_{0,\varepsilon}\delta(t')\|^2_{L^2}dt'.\end{aligned}$$

Gronwall lemma together with (4.4) gives

$$\begin{aligned}\|S^v_{0,\varepsilon}\delta(t)\|^2_{L^2} + \frac{\nu}{2}\int_0^t \|\nabla_h S^v_{0,\varepsilon}\delta(t')\|^2_{L^2}dt' &\leq \frac{C}{\nu}(\varepsilon+1)C^4_{12}(T)\\ &\times \exp C\int_0^t\Big(1+\frac{1}{\nu^3}\left(\|u_1\|^4_{L^4_h(L^\infty_v)} + \|u_2\|^4_{L^4_h(L^\infty_v)}\right)\Big)dt'\end{aligned}$$

and thus by Lemma 2.4

$$\|S^v_{0,\varepsilon}\delta(t)\|^2_{L^2} + \frac{\nu}{2}\int_0^t \|\nabla_h S^v_{0,\varepsilon}\delta(t')\|^2_{L^2}dt' \leq \frac{C}{\nu}(\varepsilon+1)C^4_{12}(T)\exp C\Big(1+\frac{1}{\nu^3}C^4_{12}(T)\Big).$$

Passing to the limit when ε tends to 0 allows to conclude the proof of Assertion (4.3). ■

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