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A Transmission Strategy for Hyperbolic Internal Waves of Small Width

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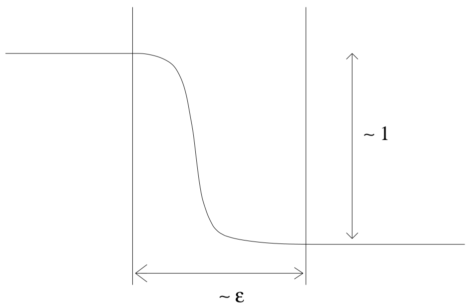
Abstract. Semilinear hyperbolic problems with source terms piecewise smooth and discontinuous across characteristic surfaces yield similarly piecewise smooth solutions. If the discontinuous source is replaced with a smooth transition layer, the discontinuity of the solution is replaced by a smooth internal layer. In this paper we describe how the layer structure of the solution can be computed from the layer structure of the source in the limit of thin layers. The key idea is to use a transmission problem strategy for the problem with the smooth internal layer. That leads to an *ansatz* different from the obvious candidates. The obvious candidates lead to overdetermined equations for correctors. With the transmission problem strategy we compute infinitely accurate expansions.

§1. Introduction.

We study internal waves of width $\varepsilon \to 0$ separating values $\overline{U}^+ \neq U^-$ on two sides. In the special case of $\overline{U}^+ = \overline{U}^-$ one has a pulse of width ε.



Consider the system of partial differential operators

$$L(t,x,\partial) \;=\; \partial_t + \sum A_j(t,x)\,\partial_j + B(t,x)\,, \qquad \partial_j \;:=\; \frac{\partial}{\partial x_j}\,,$$

where A_j, B are $N \times N$ complex matrix valued functions satisfying

$$\partial^\alpha_{t,x}\{A_j, B\} \;\in\; L^\infty(\mathbb{R}_t \times \mathbb{R}^d_x)\,.$$

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The principal symbol and characteristic variety are,

$$L_1(t,x,\tau,\xi) \;:=\; \tau I + \sum A_j(t,x)\,\xi_j\,,$$

and

$$Char(L) \;:=\; \big\{\det L_1(t,x,\tau,\xi) = 0\,\big\}\,.$$

Assumption 1. *The system L is strictly hyperbolic, or symmetric hyperbolic.*

Assumption 2. $\Sigma := \{x_d = 0\}$ *is a characteristic hypersurface for L. Furthermore, on a conic neighborhood of the conormal variety $\mathcal{N}^*\Sigma$, Char(L) is a smooth embedded hypersurface $\tau = \tau(t,x,\xi)$ in $\mathbb{R}^{2(1+d)}_{(t,x,\tau,\xi)}$.*

Examples. Assumption 2 is satisfied in the following situations. **i.** L is strictly hyperbolic,
ii. Generic characteristic hyperplanes when L_1 has constant coefficients,
iii. Symmetric systems with characteristics of constant multiplicity, hence for Maxwell and linearized compressible Euler.

We consider semilinear equations with smooth nonlinearity

$$G \;\in\; C^\infty\big(\mathbb{C}^N\,;\,\mathbb{C}^N\big)\,, \qquad G(0)=0,\;\; G'(0)=0\,.$$

Main Problem. *Describe the behavior of solutions u^ε to*

$$L\,u^\varepsilon + G(u^\varepsilon) \;=\; f^\varepsilon\,, \qquad u^\varepsilon = f^\varepsilon = 0 \quad \text{when} \quad t<0\,.$$

where

$$f^\varepsilon \;=\; F(t,x,x_d/\varepsilon)\,,$$

with $F(t,x,z)$ smooth, compactly supported in x, with limits

$$\lim_{\pm z\to\infty} F(t,x,z) \;=\; \overline{F}^{\pm}(t,x)$$

rapidly achieved.

The source term f^ε has an internal layer in an ε neighborhood of $x_d=0$. The amplitude is chosen so that there are nonlinear effects in the leading term, and there is existence on an ε independent interval. The problem is to describe the resulting internal layer in the family u^ε.
Define a discontinuous piecewise smooth function

$$\overline{f}(t,x) \;:=\; \overline{F}^{\pm}(t,x)\,, \quad \text{when} \quad \pm\,x_d > 0\,,$$

As $\varepsilon \to 0$, $f^\varepsilon \to \overline{f}$.

The limit $\varepsilon \to 0$ yields (the outer problem),

$$L\bar{U} + G(\overline{U}) \;=\; \bar{f}\,, \qquad \overline{U} = \bar{f} = 0 \quad \text{for} \quad t < 0\,.$$

The limit source, $\bar{f}$, is piecewise smooth and discontinuous across the characteristic surface Σ. It follows ([RR2], [M1,M2]) that there exists a unique local in time piecewise smooth solution, $\overline{U} \in L^\infty([0,T_1] \times \mathbb{R}^d)$. Denote by $\overline{U}^\pm$ the restriction to $\pm x_d > 0$.

Since $\overline{U}$ jumps and u^ε does not, the convergence $u^\varepsilon \to u$ is not uniform. The problem is to find correctors to $\overline{U}$ to describe u^ε with error uniformly small. Equivalently, from the detailed structure of the transition layer in f^ε, predict the details of transition layer for u^ε.

We follow a familiar two step process.

Step 1. *Find an ansatz yielding an approximate solution $u^\varepsilon_{\text{approx}}$ with small residual.* In our case the residual will have conormal (to Σ) derivatives and $(\varepsilon\partial_{t,x})^\alpha$ derivatives in $\{x_d \neq 0\}$ of size $O(\varepsilon^N)$ for all N.

Step 2. *A nonlinear stability theorem shows that the difference between the exact and approximate solutions is $O(\varepsilon^\infty)$.* In the present case, this is a known stability result ([G2], [RK]). The key is constructing approximate solutions.

Warning. Concerning the first step, the obvious *ansatz* motivated by the cases of wave trains and short pulses yields overdetermined equations for correctors to the leading approximation. This is so even in the linear case.

The goal of these notes is to motivate and describe the transmission strategy which we employ. This strategy has been effective in related problems with layers coming from a vanishing viscosity limit [GMWZ], [Sueur].

§2. The obvious ansatz fails.

A typical expansion for **linear wave packets** has the form

$$e^{ix_d/\varepsilon}\Big(a_0(t,x) + \varepsilon a_1(t,x) + \cdots\Big)\,.$$

Nonlinear wave packets include harmonics, and the possibility of adjusting the amplitude

$$\varepsilon^p\Big(U_0(t,x,x_d/\varepsilon) + \varepsilon U_0(t,x,x_d/\varepsilon) + \cdots\Big), \qquad U_j(t,x,\theta) \;\text{ periodic in } \theta.$$

A simple **short pulse** is,

$$f(t,x,x_d/\varepsilon)\,, \qquad f(t,x,\pm\infty) = 0\,.$$

The obvious *ansatz* for **nonlinear short pulses** is

$$\varepsilon^p\Big(U_0(t,x,x_d/\varepsilon) + \varepsilon U_1(t,x,x_d/\varepsilon) + \cdots\Big), \qquad U_j(t,x,\pm\infty) = 0.$$

A simple **internal wave** has the form

$$f(t,x,x_d/\varepsilon)\,,\qquad f(t,x,\pm\infty)\ \text{ exist}\,.$$

The obvious *ansatz* for **nonlinear internal waves** is

$$\varepsilon^p\Big(U_0(t,x,x_d/\varepsilon)+\varepsilon U_0(t,x,x_d/\varepsilon)+\cdots\Big),\qquad U_j(t,x,\pm\infty)\ \text{ exist.}$$

For our problem, $p=0$ is the critical power for nonlinear affects in the leading asymptotics. Plugging the *ansatz* into the equation. collecting terms according to powers of ϵ and setting the coefficients equal to zero yields equations for the U_j which **look like** you can solve them one after the other.

However, the equations for the corrector U_1 are overdetermined for pulses and internal waves. The equations have parts determined by equations of the form

$$\frac{\partial g}{\partial x_d}\ =\ f\,,\qquad g(-\infty)=a,\quad g(+\infty)=b$$

This boundary value problem is overdetermined. Solvability requires the moment condition,

$$b-a\ =\ \int_{-\infty}^{\infty} f(x_d)dx_d.$$

Even for linear problems, this is not satisfied generically.

That is the bad news. The good news is that we will find approximate solutions with error $O(\varepsilon^N)$ for all N. So we know what the solution looks like.

For the special case of pulses where the leading term is known accurate [AR], we provide approximations of order ε^∞ which are new.

Future direction. We want to understand the behavior of internal and boundary layers for times $O(1/\varepsilon)$ where diffractive effects parallel to the boundary will likely be present. Stay tuned.

§**2. Main result.**

Assumption 1 and 2 imply that $\dim\ker L_1(t,x,\tau(t,x,\xi),\xi)$ is constant for (t,x,ξ) in a conic neighborhood of $\mathcal{N}^*(\Sigma)$. In particular, $\dim\ker A_d(t,x',0)=k$ is constant on Σ.

By an t,x-dependent change of basis (orthogonal in the symmetric case), we can assume that

$$A_d(t,x',0)\ =\ \begin{pmatrix}0_{k\times k} & 0_{k\times N-k}\\ 0_{N-k\times k} & \mathcal{A}(t,x')\end{pmatrix}\,,$$

$$\det\mathcal{A}(t,x')\ \geq\ \delta>0\,,$$

$$\underline{\pi}\ :=\ \begin{pmatrix}I_{k\times k} & 0_{k\times N-k}\\ 0_{N-k\times k} & 0_{N-k\times N-k}\end{pmatrix}\,.$$

Define the **group velocity**

$$\mathbf{v}(t,x') \ := \ -\nabla_\xi \tau(t,x'\,,\, x_d=0, \tau=0\,,\, \xi'=0\,,\, \xi_d=1)\,.$$

Since τ vanishes on $\mathcal{N}^*\Sigma$ and is homogeneous of degree 1, it follows that $\mathbf{v}$ is tangent to $\{x_d=0\}$.
The algebraic lemmas of geometric optics show that the differential operator

$$\underline{\pi} L(t,x'\,,\, x_d=0\,,\, \partial)\underline{\pi}$$

is essentially a directional derivative,

$$\underline{\pi}\ L(t,x',0,\partial)\ \underline{\pi}\ \ =\ \ \underline{\pi}\big(\partial_t + \mathbf{v}(t,x').\partial'_x\big)\ +\ \text{lower order terms}\,.$$

The analogous transport operator for internal waves, $\mathbb{H}$, is

$$\begin{aligned}\mathbb{H} = \underline{\pi}\,\big(\,\partial_t + \mathbf{v}(t,x').\partial_{x'} + \partial_d\tau(t,x',0;0,\dots,0,1)z\partial_z\,\big)\\ + \text{ lower order terms}\end{aligned}$$

In $\pm z\geq 0$, define $\widetilde{F}_0^\pm(t,x',z)$ with $\tilde{F}^\pm(t,x',\pm\infty)=0$ by

$$\widetilde{F}_0^\pm(t,x',z)\ \ :=\ \ F(t,x'\,,\, x_d=0\,,\, z) - \overline{F}^\pm(t,x',x_d=0)\,.$$

Denote by

$$\mathcal{Z}\ \ :=\ \ (\partial_t,\partial_1,\dots,\partial_{d-1},\phi(x_d)\partial_d)$$

the standard conormal derivatives.

Main Theorem. *Define in $\{\pm x_d\geq 0\}\times\{\pm z\geq 0\}$ the principal profile*

$$U_0^\pm\ \ :=\ \ \overline{U}^\pm(t,x) + \widetilde{U}_0^\pm(t,x',z)\,,$$

where $\widetilde{U}_0^\pm\big(t,x',z\big)\in H^\infty\big([0,T_2]\times\mathbb{R}^{d-1}\times\mathbb{R}\big)$ is determined as the local solution of,

$$(I-\underline{\pi})\widetilde{U}_0^\pm = 0\,,\qquad \widetilde{U}_0^\pm\big|_{t<0}\ \ =\ \ 0\,,$$

$$\mathbb{H}\,\widetilde{U}_0^\pm + \underline{\pi}\Big(G(\overline{U}_0^\pm|_{x_d=0} + \widetilde{U}_0^\pm) - G(\overline{U}_0^\pm)\Big) = \underline{\pi}\widetilde{F}_0^\pm\,.$$

Then $u^\varepsilon - U_0(t,x,x_d/\varepsilon) = O(\varepsilon)$ in the sense that if ε is sufficiently small then u^ε exists on $[0,T_2]$ and $\forall\beta$,

$$\Big\|\big(\mathcal{Z},\varepsilon\partial_d\big)^\beta\Big(u^\varepsilon - U_0(t,x,x_d/\varepsilon)\Big)\Big\|_{L^\infty([0,T_2]\times\mathbb{R}^d_\pm)} = O(\varepsilon)$$

Remarks. i. If coordinates are chosen so that the hyperplanes x_d = const. are all characteristic then the $z\partial_z$ term is not present in $\mathbb{H}$. **ii.** We construct approximations of accuracy $O(\varepsilon^\infty)$ (see [GR]).

§3. The transmission strategy.

A hint that the moment conditions should not be a fatal stumbling block comes from the following observation. *In $U_0(t,x,z)$ one makes the substitution $z = x_d/\varepsilon$. In $x_d > 0$ only the limit at $z = \infty$ counts and in $x_d < 0$ only the limit at $z = -\infty$ counts. One never needs both $z = \pm\infty$ limits.*

To capitalize on this, it is natural to split the problem according to the two sides $\pm x_d > 0$. This leads naturally to the transmission strategy which we follow. The initial value problem for u^ε is equivalent to the transmission problem

$$L\, u^\varepsilon + G(u^\varepsilon) \;=\; f^\varepsilon \quad \text{in} \quad \{x_d \neq 0\}\,, \qquad \Big[(I - \underline{\pi})u^\varepsilon\Big]_{x_d=0} \;=\; 0\,. \tag{1}$$

The square brackets indicate the jump from $x_d = 0^-$ to $x_d = 0^+$. The transmission condition guarantees that when the functions on the two sides are glued together at $\{x_d = 0\}$, $A_d u$ will be continuous so there will be no delta functions produced when the differential operator is applied.

The *ansatz* for u^ε has profiles for each half space. A preliminary version is

$$u^\varepsilon \;=\; U^\varepsilon(t, x, x_d/\varepsilon)$$

where, $U^\varepsilon(t,x,z)$ is compactly supported in x with asymptotic expansions

$$U^\varepsilon(t,x,z) \;\sim\; \sum_{j=0}^{\infty} \varepsilon^j\, U_j^\pm(t,x,z)\,, \quad \text{in} \quad \{\pm x_d \geq 0\} \times \{\pm z \geq 0\}$$

$$U_j^\pm(t,x,z) \;=\; \overline{U}_j^\pm(t,x) + \widetilde{U}_j^\pm(t,x,z)\,,$$

with $\widetilde{U}_j^\pm$ rapidly decreasing as $\pm z \to \infty$. We do **not** require that $\widetilde{U}^\pm \to 0$ when $z \to \mp\infty$. In fact, $\widetilde{U}^\pm$ is not even defined at such points.

At the heart of our analysis is a calculus of such expansions. The first remark it that, *without loss of generality, the $\tilde{U}_j$ parts can be taken independent of x_d.* In fact, because of the rapid decrease, $\widetilde{U}_j(t,x,x_d/\varepsilon)$ is essentially supported in an ε neighborhood of $x_d = 0$. Taylor expansion in x_d yields

$$\widetilde{U}_j^\pm(t,x',x_d,z) \;\sim\; \sum_{k=0}^{\infty} \frac{x_d^k}{k!}\, \partial_{x_d}^k \widetilde{U}_j^\pm(t,x',0,z)\,.$$

Replacing x_d by εz yields an equivalent profile with the property that the z dependent parts depend only on t, x', z and not on x_d.

This leads to the final form for the *ansatz* where

$$U_j^{\pm}(t,x,z) \;=\; \overline{U}_j^{\pm}(t,x) + \widetilde{U}_j^{\pm}(t,x',z)$$

with $\widetilde{U}_j^{\pm}$ independent of x_d and rapidly decreasing as $\pm z \to \infty$. Such expansions are unique.

Proposition 1. *If a family u^ε has an asymptotic expansion of this form, then the profiles $\overline{U}_j^{\pm}$ and $\widetilde{U}_j^{\pm}$ are uniquely determined.*

A different way to generate smoothed sources f^ε is to take a standard mollification of the piecewise smooth source $\overline{f}$. Suppose that $j(t,x)$ is smooth compactly supported in $t \geq 0$ with $\int j\, dt\, dx = 1$. Define $j^\varepsilon(t,x) = \varepsilon^{-d-1} j(t/\varepsilon, x/\varepsilon)$. Denote by J^ε the operator which is convolution with j^ε.
Suppose that $\overline{f}$ is piecewise smooth and compactly supported on on $\{t \leq T\} \times \mathbb{R}^d$ with jumps on $\{x_d = 0\}$.

Proposition 2. *With the hypotheses of the preceding paragraph, $f^\varepsilon := J^\varepsilon \overline{f}$ has an asymptotic expansion of the above form.*

Proposition 3. *The set of families u^ε which have expansions of the form is invariant under smooth change of coordinates*

$$\big(\tilde t, \tilde x\big) = \big(\tilde t(t,x), \tilde x(t,x)\big)\,, \qquad \big(t,x\big) = \big(t(\tilde t,\tilde x), x(\tilde t,\tilde x)\big)$$

which map the half spaces $\pm x_d > 0$ to the corresponding halfspaces $\pm \tilde x_d > 0$.

This is crucial when one wants to study characteristic surfaces which are not presented in the form $\{x_d = 0\}$. The first step is to flatten the surface and Proposition 3 shows that the families with expansions are independent of the flattening.

Proposition 4. *If u^ε has an expansion of the above form and u^ε satisfies the transmission condition in (1), then $Lu^\varepsilon + G(u^\varepsilon)$ has an expansion*

$$Lu^\varepsilon + G(u^\varepsilon) \;=\; W^\varepsilon(t,x,x_d/\varepsilon) \;\sim\; \sum_{j=-1}^{\infty} \varepsilon^j\, W_j(t,x,x_d/\varepsilon)$$

where W_j is compactly supported in x and smooth in $\pm x_d \geq 0, \pm z \geq 0$, and

$$W_j(t,x,z) \;=\; \overline{W}_j^{\pm}(t,x) + \widetilde{W}_j^{\pm}(t,x',z)$$

with $\widetilde{W}_j^{\pm}(t,x',z)$ rapidly decreasing as $\pm z \to \infty$. For each $n \geq 0$ the profiles W_j with $j \leq n-1$ are given by explicit formulas involving terms in the expansion of u^ε up to an including $O(\varepsilon^n)$.

Remarks. 1. The expansion starts with an ε^{-1} term. **2.** If the transmission condition were not exactly satisfied there would be $\delta(x_d)$ terms on the left from ∂_d applied to a jump.

The approximate solution is constructed by injecting the *ansatz* and evaluating as in Proposition 4. Setting the successive terms equal to zero yields determined (and not overdetermined) equations for the profiles U_j. The computation is interesting in detail and resembles similar computations for boundary layers. The details can be found in the preprint on my web page. The recipe for U_0 is described in the Main Theorem.

For the transmission strategy, some components of the $\tilde{U}_j^{\pm}$ are determined from differential equations of the form

$$\begin{aligned} \partial_z g^+ &= f^+ \quad \text{in} \quad z > 0\,, \qquad g^+(+\infty) = 0\,, \\ \partial_z g^- &= f^- \quad \text{in} \quad z < 0\,, \qquad g^+(-\infty) = 0\,. \end{aligned}$$

The problematic moment conditions have as a consequence that the $g^{\pm}$ do not match continuously at $z = 0$. However, this is **not** an obstruction to the smoothness of u^ε since the $\bar{U}$ terms are also discontinuous, and what is important is the sum.

Summary. Once the transmission problem form of the *ansatz* and the basic calculus is in hand, the proofs are interesting in detail. The stability is known. The key is finding the *ansatz* and the key step in the discovery step is the fact that for the obvious *ansatz*, the two limits $U(t, x, \pm\infty)$ never occur for the same point (t, x) with $x_d \neq 0$. There is a natural decoupling of $\pm x_d > 0$, which leads to the transmission problem approach.

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