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Résumé: Nous discutons l’asymptotique des noyaux de Bergman pour des puissances élevées de fibrés de droites, d’après deux travaux récents avec B.Berndtsson et R. Berman*

0. Introduction.

We present some new proofs and results around the so called Tian–Yau–Zelditch–Catlin asymptotics for the orthogonal projections onto the spaces of harmonic forms with coefficients in a high power of a complex line-bundle:

1) For (0,0)-forms: Here we give a new proof (joint work with B.Berndtsson and R.Berman [BeBeSj].)

2) For (0,q)-forms: New result and proof (joint work with R.Berman [BeSj]).

The subject has gained new interest recently through the work of geometers. M. Shubin suggested closely related problems to me 11 years ago, and later I got more stimulation mainly through the works of Shiffman, Zelditch and coworkers and from discussions with Berndtsson and Berman around the work [Be], as well as with X.Ma. The plan of the talk is:

1) Statement of the result.
2) Some historical remarks.
3) Quick outline of a new proof for (0,0)-forms.
4) Outline of the proof for (0,q)-forms.

1. The result

Let \( L \) be a holomorphic line bundle over a complex compact manifold \( X \) of dimension \( n \). Assume the fibers \( L_x \) and \( \wedge^{1,0} T^*_x X \) carry Hermitian metrics that depend smoothly on \( x \in X \).

If \( s \) is a non-vanishing holomorphic section of \( L \) on the open subset \( \tilde{X} \subset X \), write \( |s(x)| = e^{-\phi(x)} \) with \( \phi(x) \) real and smooth. The curvature form of the line bundle is then determined by

\[
\bar{\partial} \bar{\partial} \phi = \sum \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k,
\]

where the right hand side is written in local holomorphic coordinates. Assume that \( \bar{\partial} \bar{\partial} \phi \) is non-degenerate of constant signature \((n_+, n_-)\) on \( X \).

We shall replace \( L \) by \( L^k \) and consider the \( \bar{\partial} \)-complex:

\[
C^\infty(X; L^k \otimes \wedge^{0,0} T^* X) \rightarrow C^\infty(X; L^k \otimes \wedge^{0,1} T^* X)
\]

\[
\rightarrow \cdots \rightarrow C^\infty(X; L^k \otimes \wedge^{0,n} T^* X).
\]

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If we also fix a positive smooth integration density $m(dx)$, we have the adjoint $\bar{\partial}^\dagger$-complex

$$C^\infty(X; L^k \otimes \wedge^{0,0} T^* X) \leftarrow C^\infty(X; L^k \otimes \wedge^{0,1} T^* X) \leftarrow \ldots \leftarrow C^\infty(X; L^k \otimes \wedge^{0,n} T^* X).$$

We introduce

$$h = 1/k \ll 1, \text{ for } k \gg 1$$

and the Hodge Laplacian for $(0,q)$-forms:

$$\Delta_q = \Delta_{q,k} = h^\partial h^{\partial^*} + h^{\partial^*} h^\partial.$$ (1.3)

$X$ being compact, $\Delta_q$ is essentially self-adjoint with discrete spectrum contained in $[0, +\infty[$. Let $\mathcal{N}(\Delta_q)$ be the kernel (i.e. the 0-eigenspace) and let

$$\Pi_q : L^2(X, L^k \otimes \wedge^{0,q} T^* X) \to \mathcal{N}(\Delta_q)$$

be the orthogonal (Bergman) projection. With $\tilde{X}$, $s$, $\phi$ as above, we have the unitary identifications

$$L^2(\tilde{X}; \wedge^{0,q} T^* X) \leftrightarrow L^2(\tilde{X}; L^k \otimes \wedge^{0,q} T^* X)$$

$$u \leftrightarrow (e^\phi s)^k u$$

$$Z_\phi \leftrightarrow h\bar{\partial}$$

$$\Delta_{q,\text{loc}} \leftrightarrow \Delta_q$$

$$\Pi_{q,\text{loc}} \leftrightarrow \Pi_q,$$

with

$$Z_\phi = (e^\phi s)^{-k} \circ h\bar{\partial} \circ (e^\phi s)^k = h\bar{\partial} + (\bar{\partial} \phi)^\wedge,$$

$$\Delta_{q,\text{loc}} = Z_\phi^* Z_\phi + Z_\phi Z_\phi^*,$$

$$\Pi_{q,\text{loc}} = (e^\phi s)^{-k} \Pi_q (e^\phi s)^k.$$ (1.4)

For the proof in the case of $(0,0)$-forms we shall also use the unitary identification

$$L^2(\tilde{X}; \wedge^{0,0} T^* X, e^{-\frac{2\phi}{h}} m) \leftrightarrow L^2(\tilde{X}; L^k \otimes \wedge^{0,0} T^* X)$$

$$e^{\phi/h} u \leftrightarrow (e^\phi s)^k u.$$ (1.5)

$\Delta_{q,\text{loc}}$ has a scalar principal symbol $p \geq 0$ (times the identity matrix) vanishing precisely to the second order on the symplectic submanifold $\Sigma \subset T^* X$, given by

$$\zeta = \frac{2}{i} \frac{\partial \phi}{\partial z}, \ z = x + iy, \ \zeta = \xi - i\eta,$$

with $(x, y; \xi, \eta)$ as standard canonical coordinates on $T^* X$ (and $z = (z_1, \ldots, z_n)$ denoting local holomorphic coordinates).

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In [MeSj1] and later in [BoGu] it was established that there exist almost analytic manifolds (in the sense of [MelSj]) and we shall from now on use the term almost holomorphic

\[ J_+, J_- \subset T^* X^\mathbb{C}, \quad J_- = \overline{J}_+, \]

such that \( J_+ \cap J_- = \Sigma^\mathbb{C} \) with transversal intersection, such that locally

\[ J_+ : f_1 = \ldots = f_n = 0, \quad \{ f_j, f_k \}_{|J_+} = 0, \]

\[ (\frac{1}{i} \{ f_j, f_k \})_{j,k} > 0 \text{ on } \Sigma, \quad p_{|J_+} = 0. \]

When \( n_- = 0 \), we can take \( f_j \) to be the semi-classical symbol of \( h \frac{\partial}{\partial x_j} + \frac{\partial \phi}{\partial x_j} \) that will be given more explicitly below. The following theorem is mainly due to S.Zelditch and D.Catlin when \( q = n_- = 0 \) and to R.Berman and Sjöstrand in the general case.

**Theorem 1.1.** For \( k = 1/h \) sufficiently large, we have \( \Pi_q = 0, \quad q \neq n_- * \) and for \( q = n_- \):

\[
\Pi_{q, \text{loc}} u(x) = h^{-n} \int e^{\frac{1}{h} \psi(x,y)} b(x,y) u(y)m(dy) + Ru,
\]

for \( x \in \tilde{X}, \ u \in L^2(\tilde{X}, L^k \otimes \wedge^{0,q} T^* X), \) where \( b \sim \sum_0^\infty b_j(x,y)h^j \) in \( C^\infty(\tilde{X} \times \tilde{X}; L(\wedge^{0,q} T^*_y X, \wedge^{0,q} T^*_x X)) \),

\[ Ru = \int r(x,y) u(y)m(dy), \quad \partial_x^a r = O(h^\infty). \]

Further, \( \psi(x,x) = 0, \ Re \psi(x,y) \sim -\text{dist}(x,y)^2, \)

\[
\left\{ \begin{array}{l}
(x, d_x \frac{1}{i} \psi(x,y)) \in J_+ \mod O(\text{dist}(x,y)\infty).
\end{array} \right.
\]

For \( x = y \):

\[ \frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x}, \quad \frac{\partial \psi}{\partial \overline{x}} = -\frac{\partial \phi}{\partial \overline{x}}, \quad \frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial y}, \quad \frac{\partial \psi}{\partial \overline{y}} = \frac{\partial \phi}{\partial \overline{y}}. \]

**2. Historical remarks.**

Most of the earlier results concern the positively curved case \( n_- = 0 \). G.Tian [Ti], followed by W.Ruan [Ru] and Z.Lu [Lu], computed increasingly many terms of the asymptotic expansion on the diagonal, using Tian’s method of peak solutions. T. Bouche [Bou] also got the leading term using heat kernels.

S.Zelditch [Ze], D.Catlin [Ca] established the complete asymptotic expansion at \( x = y \) by using a result of Boutet de Monvel, Sjöstrand [BoSj] for the asymptotics of the Szegö kernel on a strictly pseudoconvex boundary (after the pioneering work of C.Fefferman [Fe]), here on the boundary of the unit disc bundle, and a reduction idea of Boutet de Monvel, Guillemin [BoSj]. Scaling asymptotics away from the diagonal was obtained later

* as follows from Hörmander’s \( L^2 - \overline{\partial} \) estimates [Hö].

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by P. Bleher, B. Shiffman, Zelditch [BlShZe] and the full asymptotics by L. Charles [Ch],
using again the reduction method.

In more general situations, full asymptotic expansions on the diagonal and in some
sense away from the diagonal were obtained by X. Dai, K. Liu, X. Ma [DaLiMa] (see also
[MaMar] for related spectral results).

Without a positive curvature assumption there are fewer results. J. M. Bismut [Bi] used
the heat kernel method in his approach to Demailly’s holomorphic Morse inequalities. X.
Ma has pointed out to us that the method and results of [DaLiMa] can be extended to
the case of non-positive holomorphic line bundles by using a spectral gap estimate from
[MaMar].

3. Quick outline of a proof when \( q = n_− = 0 \) ([BeBeSj])

Locally, the problem is essentially to find the orthogonal projection from \( L^2(\mathbb{C}^n, e^{−2φ/h}m(dx)) \)
to its subspace of holomorphic functions. That projection was recently constructed in
[MeSj3], and the method we present here is similar but differs on one essential point: A
square root procedure is replaced by a simpler algorithm. Write for \( u ∈ L^2_φ ∩ \text{Hol} \):

\[
1u(x) = \frac{1}{(2πh)^n} \int_{Γ(x)} e^{\frac{θ}{h}(x−y)} u(y)dydθ
\]

modulo an error \( O(h^∞) \), provided that the symbol \( a ≈ \sum_0^∞ a_j h^j \) (is almost holomorphic
at a suitable set and) satisfies

\[
\sum_{α ∈ \mathbb{N}^n} \frac{h^{||α||}}{α!} (\partial_α^y D_y^α a(x, y, θ; h))_{y=x} \sim 1.
\] (3.2)

Let \( Ψ(x, y), M(x, y) \) be almost holomorphic with \( Ψ(x, x) = φ(x), M(x, x) = m(x) \). Recall
that in the case \( n_− = 0 \), \( φ \) is strictly plurisubharmonic and we have

\[-φ(x) + 2\text{Re} Ψ(x, y) − φ(y) \sim −|x − y|^2.\]

Consider

\[
Ju(x) = \int_{Γ(x)} e^{\frac{θ}{h}(Ψ(x, w)−Ψ(y, w))} c(x, w; h)M(y, w)u(y) \frac{dydw}{h^n}
\]

\[
= \int_{Γ(x)} e^{\frac{θ}{h}Ψ(x, y)} c(x, y; h)u(y)e^{−\frac{θ}{h}φ(y)} m(y) \frac{dydy}{h^n}
\]

where we integrate over \( w = y \) in the first integral.

Use the Kuranishi trick:

\[
2(Ψ(x, w) − Ψ(y, w)) = i(x − y) ⋅ θ(x, y, w),
\]

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\[ J u(x) = \int \int e^{i \frac{h}{2}(x-y) \cdot \theta} a(x, y, \theta; h) u(y) \frac{dy d\theta}{(2\pi h)^n}, \]

\[ a(x, y, \theta; h) = (2\pi)^n c(x, w(x, y, \theta); h) M(y, w(x, y, \theta)) \det \left( \frac{\partial \theta}{\partial w} \right). \]

Here the coefficients \( c_0, c_1, \ldots \) in the asymptotic expansion of \( c \) can be determined successively so that (3.2) holds.

**4. Outline of the proof for general \( n \) ([BeSj])**

We shall use the heat equation approach of [MeSj] with a Witten complex trick. Work locally with \[ \Delta_{q, loc} = Z_\phi^* Z_\phi + Z_\phi Z_\phi^*. \]

Let \( x_1, \ldots, x_{2n} \) be local coordinates. Construct a parametrix \( U_q(t; h) \) for

\[ (h \partial_t + \Delta_{q, loc}) U_q(t) = O(h^\infty), \quad U_q(0) = id, \quad U_q(t) u(x) = \int \int e^{\frac{i}{h} (\psi(t, x, \eta) - y \eta)} a(t, x, \eta; h) u(y) \frac{dy d\eta}{(2\pi h)^{2n}}. \]

Here we can solve

\[ i \partial_t \psi + p(x, \psi' x) = O((\text{Im} \psi)'^\infty), \]

locally with \( \psi(0, x, \eta) = x \cdot \eta \) and with \( \text{Im} \psi \geq 0 \), and more precisely

\[ \text{Im} \psi \sim \text{dist}(x, \eta; \Sigma)^2, \quad t \geq t_0 > 0, \]

\[ \psi(t, x, \eta) = x \cdot \eta + O(\text{dist}(x, \eta; \Sigma)^2) \]

(See [MelSj2] and references given there to work of Kucherenko and others.) In [MeSj] a more detailed study was given, using that \( \Sigma \) is symplectic, and we showed that there exists a limiting function \( \psi(\infty, x, \eta) \) such that

\[ \partial^\alpha_{t, x, \eta} (\psi(t, x, \eta) - \psi(\infty, x, \eta)) = O_\alpha(1) e^{-t/C}, \quad (4.3) \]

for \( t \geq 0 \), \( (x, \eta) \in \Sigma \). As used in [MeSj1,2], \( J_\infty \) can be viewed as the stable outgoing and incoming manifolds for the \( i^{-1} H_p \) flow around the fixed point variety \( \Sigma^C \), and the canonical transformation \( \kappa_t \) generated by \( \psi(t, \cdot, \cdot) \) converges to the limiting canonical relation \( \kappa_\infty \) characterized by saying that \( (\rho, \mu) \in \text{graph}(\kappa_\infty) \) if \( \rho \in J_+, \mu \in J_- \) belong to bicharacteristics leaves of \( J_+, J_- \) respectively, containing the same point of \( \Sigma^C \).

The symbol

\[ a(t, x, \eta; h) \sim \sum_0^\infty a_j(t, x, \eta) h^j, \]

is determined by a sequence of transport equations, and adapting the approach of [MeSj1] to the case of matrix-valued symbols, we get on \( \Sigma \):

\[ \partial^\alpha_{t, x, \eta} a_j = \begin{cases} O_{\alpha, j}(1) e^{-t/C}, & q \neq n_- \\ O_{\epsilon, \alpha, j}(1) e^{\epsilon t}, & \forall \epsilon > 0, \quad q = n_- \end{cases}, \quad (4.4) \]

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Now, let $q = n_-$ and apply a Witten trick: From
\[ \Delta_{q+1,\text{loc}} Z_{\phi} = Z_{\phi} \Delta_{q,\text{loc}}, \quad \Delta_{q-1,\text{loc}} Z_{\phi}^* = Z_{\phi}^* \Delta_{q,\text{loc}}, \]
we get
\[
(h \partial_t + \Delta_{q-1,\text{loc}}) Z_{\phi} U_q(t) = O(h^\infty),
\]
\[
(h \partial_t + \Delta_{q+1,\text{loc}}) Z_{\phi}^* U_q(t) = O(h^\infty).
\]
Here $Z_{\phi} U_q, Z_{\phi}^* U_q$ have the general form (4.2) and since $q - 1 \neq n_- \neq q + 1$, one can show that the symbols satisfy the same decay estimate as in the first case in (4.4).

This also applies to
\[
\Delta_{q,\text{loc}} U_q = Z_{\phi}(Z_{\phi}^* U_q) + Z_{\phi}^*(Z_{\phi} U_q),
\]
and by (4.1) to
\[
\frac{h}{\partial U_q(t)} = \int \int e^{\frac{i}{h}(\psi(t,x,\eta) - y \eta)} (i \frac{\partial \psi}{\partial t} u + h \frac{\partial a}{\partial t} u(y)) \frac{dyd\eta}{(2\pi h)^{2n}}.
\]
This and (4.3) imply
\[
\partial_{t,x,\eta}^\alpha a_j(t,x,\eta) = O(e^{-t/C}), \quad (x,\eta) \in \Sigma.
\]
Hence, there exists a symbol $a_j(\infty, x, \eta)$ such that on $\Sigma$:
\[
\partial_{t,x,\eta}^\alpha (a_j(t,x,\eta) - a_j(\infty, x, \eta)) = O(e^{-t/C}).
\]
We get the approximate null-projection:
\[
\Pi_{q,\text{loc}}^\approx u(x) = \int \int e^{\frac{i}{h}(\psi(\infty,x,\eta) - y \eta)} a(\infty, x, \eta; h) u(y) \frac{dyd\eta}{(2\pi h)^{2n}}
\]
\[
= \int e^{\frac{i}{h}\psi_{\text{new}}(x,y)} b(x, y; h) u(y) \frac{m(dy)}{h^n} + Ru,
\]
where the last equality follows from complex stationary phase ([MelSj]) and the last expression is as in the theorem.

The remaining part is more routine. We get:
\[
U_q(t) = \Pi_{q,\text{loc}}^\approx + V_q(t),
\]
\[
V_q(t) = O(e^{-t/C}) : H_{-\infty}^{\text{comp}} \rightarrow H_{\text{loc}}^\infty, \quad t \geq t_0 > 0.
\]
Modulo $O(h^\infty)$:
\[
\Delta_{q,\text{loc}} \Pi_{q,\text{loc}}^\approx \equiv \Pi_{q,\text{loc}}^\approx \Delta_{q,\text{loc}} \equiv 0,
\]
\[
(\Pi_{q,\text{loc}}^\approx)^* \equiv \Pi_{q,\text{loc}}^\approx,
\]
\[
[\Pi_{q,\text{loc}}^\approx, V(t)] = O(e^{-t/C} h^\infty).
\]

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Approximate resolvent for Re \( z < (2C)^{-1} \), \( |z| \geq h^{N_0} \):

\[
R_{\text{loc}}^{\approx}(hz) = \frac{1}{hz} \Pi_{q, \text{loc}}^{\approx} - \frac{1}{h} \int_{0}^{\infty} e^{tz} V_q(t) dt.
\]

(When \( q \neq n_- \), we have the simpler formula for Re \( z < (2C)^{-1} \):

\[
R_{\text{loc}}^{\approx}(hz) = -\frac{1}{h} \int_{0}^{\infty} e^{tz} U_q(t) dt.
\]

Notice that,

\[
\Pi_{q, \text{loc}}^{\approx} = \frac{-1}{2\pi i} \int_{|z|=r} R_{\text{loc}}^{\approx}(z) dz, \quad h^{N_0} \leq r \leq \frac{h}{2C}.
\]

Back to the global situation, we glue the different \( R_{\text{loc}}^{\approx} \) together and get \( R^{\approx}(z) : H^s(X) \to H^s(X) \) such that for Re \( z < (2C)^{-1} \), \( |z| \geq h^{N_0} \):

\[
(\Delta_q - h z) R^{\approx}(hz) \equiv R^{\approx}(hz)(\Delta_q - h z) \equiv \text{id}, \tag{4.5}
\]

which implies that

\[
(\Delta_q - h z)^{-1} \equiv R^{\approx}(hz),
\]

\[
\Pi_q = \frac{1}{2\pi i} \int_{|z|=r} (z - \Delta_q)^{-1} \, dz \equiv \frac{-1}{2\pi i} \int_{|z|=r} R^{\approx}(z) \, dz.
\]

Bibliography


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