Centre de Mathématiques Laurent Schwartz
Ecole Polytechnique

SEMINAIRE

# Equations aux Dérivées Partielles

2001-2002

Alexander Volberg

**Bellman approach to some problems in harmonic analysis**

*Séminaire É. D. P.* (2001-2002), Exposé n° XIX, 14 p.

<http://sedp.cedram.org/item?id=SEDP_2001-2002____A19_0>

U.M.R. 7640 du C.N.R.S.
F-91128 PALAISEAU CEDEX
Fax : 33 (0)1 69 33 49 49
Tél : 33 (0)1 69 33 49 99

cedram

*Article mis en ligne dans le cadre du*
*Centre de diffusion des revues académiques de mathématiques*
http://www.cedram.org/

# Bellman approach to some problems in harmonic analysis

A. VOLBERG

**Abstract :** The stochastic optimal control uses the differential equation of Bellman and its solution - the Bellman function. Recently the Bellman function proved to be an efficient tool for solving some (sometimes old) problems in harmonic analysis.

## 1 Bellman equation for stochastic optimal control.

Let us start with the problem of control of rather general stochastic process. The main reference is the book of N. Krylov "Optimal control of diffusion process", Springer 1980. Let $x^t$ be a stochastic process in $\mathbb{R}^d$ satisfying the following stochastic differential equation

$$x^t = x + \int_0^t \sigma(\alpha^s, x^s) dw^s + \int_0^t b(\alpha^s, x^s) ds \ . \tag{1}$$

Here $t$ is the time, $w^t$ is a $d_1$-dimensional Wiener process, $\sigma = \sigma(\alpha, y)$ is a $d \times d_1$ matrix, $b$ is a $d$-dimensional vector. The process $\alpha^t$ is supposed to be the control that we have to choose.

We denote by $A \subset \mathbb{R}^{d_2}$ the set of admissible controls, that is the domain where $\alpha$ runs.

The choice of stochastic process $\alpha^s$ usually (it will be also $d$-dimensional) gives us different "motions" - different solutions of (1). Of course, the questions of existence and uniqueness of solutions immediately arise, but we just assume the existence and the uniqueness.

Suppose we are given the profit function $f^\alpha(y)$ : on the trajectory $x^t$, for the time interval $[t, t + \Delta t]$, the profit is

$$f^{\alpha^t}(x^t)\Delta t + o(\Delta t) \ .$$

Therefore, on the whole trajectory we earn

$$\int_0^\infty f^{\alpha_t}(x^t)dt \ .$$

We want to choose the control $\alpha = \{\alpha^s\}$ to maximize the average profit

$$v^\alpha(x) = \mathbb{E}\int_0^\infty f^{\alpha^t}(x^t)dt + \overline{\lim_{t\to\infty}}\, \mathbb{E}(F(x^t)) \ ,$$

for the process starting at $x$. Here $F, F \geq 0$, is a bonus function - one gets it when one retires.

The optimal average gain is what is called the Bellman function for stochastic control :

$$v(x) = \sup_\alpha V^\alpha(x) \ .$$

It satisfies a well known Bellman differential equation. Bellman PDE is based on two things :

1) Bellman's principle,

2) Ito's formula.

1) *Bellman's principle* states that

$$\forall t \geq 0 \ , \ v(x) = \sup_\alpha \mathbb{E}\left[\int_0^t f^{\alpha^s}(x^s)ds + v(x^t)\right] \ .$$

Let us fix $t > 0$, and let us consider an individual trajectory. The profit for the interval $[0, t]$ is given by

$$\int_0^t f^{\alpha^s}(x^s)ds \ .$$

Suppose that the trajectory has reached the point, say $y$, at the moment $t$. Then the maximal average profit we can gain starting at the moment $t$ at $y$ is exactly $v(y)$. Indeed, since the increments of $w^s$ for $s \geq t$ do not depend on $w^\tau$, $\tau < t$, and they behave as corresponding increments after time 0, and the equation (1) is time invariant, there is no difference between starting at time 0 or at time $t$. Applying the full probability formula to take into account all possible endpoints $y = x^t$, we get exactly the Bellman principle.

2) *Itô's formula* (the version we need).

Let us fix a moment of time $s$, and a small increment $\Delta s$. We want to estimate the difference $v(x^{s+\Delta s}) - v(x^s)$. Recall that $w^s = (w_1^s, \cdots, w_d^s)^T$, denote $\Delta w_k^s = w_k^{s+\Delta s} - w_k^s$, $\Delta w^s = w^{s+\Delta s} - w^s$.

Using Taylor's formula (we think that Bellman's $v$ is smooth, which might not be the case) we will have to consider (among others) the term

$$\sum_{k=1}^{d} \frac{\partial v}{\partial x_k}(x^s) \sum_{j=1}^{d_1} \sigma_{kj}(\alpha^s, x^s)\Delta w_j^s +$$

$$+ \sum_{k=1}^{d} \frac{\partial v}{\partial x_k}(x^s) b_k(\alpha^s, x^s)\Delta s \ .$$

After averaging over the probability, the first term will vanish $-\Delta w_k^s$ are independent of $x^s$ and have zero averages.

The second term can be rewritten as $\mathbb{E}(\mathcal{L}_1^{\alpha^s}(x^s)v)(x^s)\Delta s$, where the first order differential operator $\mathcal{L}_1^{\alpha}(x)$ is given by

$$\mathcal{L}_1^{\alpha}(x) := \sum_{k=1}^{d} b_k(\alpha, x)\frac{\partial}{\partial x_k} \ .$$

Another term that will appear is

$$\frac{1}{2}\sum_{i,j} \frac{\partial^2 v}{\partial x_i \partial x_j} \left(\sum_k \sigma_{jk}\Delta w_k^s\right) (\sum_k \sigma_{ik}\Delta w_k^s) \ .$$

After averaging over the probability, only terms with $\mathbb{E}(\Delta w_k^s)^2 = \Delta s$ will not vanish. This gives rise to $\mathbb{E}\left(\mathcal{L}_2^{\alpha^s}(x^s)v\right)(x^s)\Delta s$, where the second order differential operator $\mathcal{L}_2^{\alpha}(x)$ is given by

$$\mathcal{L}_2^{\alpha}(x) := \sum_{i,j=1}^{d} a^{ij}(\alpha, x)\frac{\partial^2}{\partial x_i \partial x_j} \text{ , where}$$

$$a^{ij}(\alpha, x) = \frac{1}{2}\sum_{k=1}^{d_1} \sigma_{ik}(\alpha, x)\sigma_{jk}(\alpha, x) \ .$$

The higher order Taylor terms will give $\Delta s$ to powers greater than 1, and, obviously they sum up to $o(1)$, and can be omitted.

Gathering all together we get

$$\mathbb{E}(v(x^t)) = v(x) + \mathbb{E}\int_0^t \mathcal{L}^{\alpha^s}(x^s)v(x^s)ds \ , \tag{2}$$

where $\mathcal{L}^{\alpha}(x) = \mathcal{L}_1^{\alpha}(x) + \mathcal{L}_2^{\alpha}(x)$.

$\mathcal{L}_1^\alpha$ is called the drift.
Putting this into the Bellman principle one gets

$$0 = \sup_\alpha \left[ \int_0^t f^{\alpha^s}(x^s) ds + \int_0^t \mathcal{L}^{\alpha^s}(x^s) v(x^s) ds \right] .$$

Dividing by $t$ and letting $t$ tend to zero one gets Bellman's PDE :

$$\sup_{\alpha \in A} [\mathcal{L}^\alpha(x) v(x) + f^\alpha(x)] = 0 .$$

Of course, to justify taking the limit one has to make some assumptions, so the above presentation is just a scheme.

**Supersolutions :** We always think that $F \geq 0$ is convex. We have an "obstacle problem" in $\Omega \subseteq \mathbb{R}^d$ :

$$\begin{cases} \sup_{\alpha \in A} [\mathcal{L}^\alpha(x) v(x) + f^\alpha(x)] = 0, x \in \Omega \ , \\ f(x) \geq F(x) \ , \ x \in \Omega \ . \end{cases} \tag{3}$$

In our applications we will be more interested in supersolutions of the Bellman equation (2) :

$$\begin{cases} \sup_{\alpha \in A} [\mathcal{L}^\alpha(x) V(x) + f^\alpha(x)] \leq 0, x \in \Omega \ , \\ V(x) \geq F(x) \ , \ x \in \Omega \ . \end{cases} \tag{4}$$

**Lemma 1.1** *Let $V$ solve (4) and let $v$ be the Bellman function, then $V \geq v$ in $\Omega$.*

**Proof :** The first line of (3) states that $-\mathcal{L}^\alpha(x) V(x) \geq f^\alpha(x)$. Using (2) one gets

$$\begin{aligned} V(x) &= \mathbb{E} V(x^t) - \mathbb{E} \int_0^t (\mathcal{L}^{\alpha^s}(x^s) V)(x^s) ds \\ &\geq \mathbb{E} F(x^t) + \mathbb{E} \int_0^t f^{\alpha^s}(x^s) ds. \end{aligned}$$

Writing $\overline{\lim}_{t \to \infty}$ of both parts, we get $V(x) \geq v^\alpha(x)$. It rests to take the supremum over the control process $\alpha$. •

# 2 Harmonic analysis Bellman functions.

## 2.1 $A_\infty$ weights and associated Carleson measures. Buckley's inequality.

We call a nonnegative function on $\mathbb{R}$ an $A_\infty$ weight (dyadic $A_\infty$ weight actually) if

$$\langle w\rangle_j \le C_1 e^{\langle \log w\rangle_J} \ , \ \forall J \in \mathcal{D} \ . \tag{5}$$

Here $\mathcal{D}$ is a dyadic lattice on $\mathbb{R}$, $\langle \cdot \rangle_J$ is the averaging over $J$.

We are going to illustrate our use of Bellman function technique by a collection of examples, the first of which is the result of Buckeley [1] that can be found (along with "continuous analogs") in the paper of Fefferman-Kenig-Pipher [2].

**Theorem 1** *Let $w \in A_\infty$. Then*

$$\forall I \in \mathcal{D} \ , \frac{I}{|I|}\sum_{\substack{\ell \le I,\\ \ell\in\mathcal{D}}} \left(\frac{\langle w\rangle_{\ell_+} - \langle w\rangle_{\ell_-}}{\langle w\rangle_\ell}\right)^2 |\ell| \le C_2 \ , \tag{6}$$

*where $C_2$ depends only on $C_1$ in (5). Here $\ell_\pm$ are right and left sons of $\ell \in \mathcal{D}$.*

**Who moves ?**

$$x_1, x_2 = \langle w\rangle_J \ , \ \langle \log w\rangle_J$$

$$\alpha_1 = \langle w\rangle_{\text{son of} J} - \langle w\rangle_J \Rightarrow |\alpha_1| = \frac{1}{2}|\langle w\rangle_{J_-} - \langle w\rangle_{J_+}|$$

Function of profit can be read off (6) if one notices that $\frac{1}{|I|}\sum_{\substack{\ell\le I,\\ \ell\in\mathcal{D}}}\cdots$ is the average over the lines of life. Each line of life initiates at $I$ and then proceeds to $I_{\varepsilon_1}(\varepsilon_1 = +1$ or $\varepsilon_1 = -1)$, then to $I_{\varepsilon_1\varepsilon_2}(\varepsilon_2 = +1$ or $\varepsilon_2 = -1)$, etc.

Thus $\frac{1}{|I|}\sum_{\substack{\ell\le I,\\ \ell\in\mathcal{D}}}\cdots$ plays the role of $\mathbb{E}\int_0^\infty \cdots$. This allows us to choose the correct profit function

$$f^\alpha(x) = \frac{4\alpha_1^2}{x_1^2} \ .$$

Bonus function $F \equiv 0$ here.

Bellman equation reads now

$$\sup_{\alpha=(\alpha_1,\alpha_2)} \left[\langle d^2 v\alpha, \alpha\rangle + \frac{8\alpha_1^2}{x_1^2}\right] = 0 \tag{7}$$

to be solved in

$$\Omega = \left\{(x_1, x_2) : 1 \leq x_1 e^{-x_2} \leq c_1\right\} \tag{8}$$

with the obstacle condition

$$v(x) \geq 0 \quad \forall x \in \Omega\ . \tag{9}$$

We are not going to solve (7) - (9). Instead, we will write down a supersolution.

**Theorem 2** *The dyadic Bellman function $B^d$ is always a supersolution of an obstacle problem. In other words, always*

$$B^d(x) \geq v(x)\ ,\ x \in \Omega\ . \tag{10}$$

**Theorem 3** *Suppose there is no drift. If $\Omega$ is convex and $x \to f^\alpha(x)$ is convex, then any supersolution $V$ of the obstacle problem majorizes $B^d$ :*

$$V(x) \geq B^d(x)\ ,\ x \in \Omega\ . \tag{11}$$

In other words, **under convexity assumptions and no drift assumption**, one has

$$v \geq \inf_{\substack{V \text{is a supersolution} \\ V \geq F}} V \geq B^d \geq v \text{ in } \Omega\ . \tag{12}$$

We have to explain what is $B^d$. We are doing this for our example of Buckley estimate. It is defined the same way in each of the subsequent problems.

$$B^d(x_1, x_2) = \sup\{\frac{1}{|I|} \sum_{\substack{\ell \leq I \\ \ell \in \mathcal{D}}} \left(\frac{\langle w \rangle_{\ell+} - \langle w \rangle_{\ell-}}{\langle w \rangle_\ell}\right)^2 |\ell| :$$

$$\langle w \rangle_I = x_1\ , \langle \log w \rangle_I = x_2\ , \langle w \rangle_J \leq C_1 e^{\langle \log w \rangle} J \text{ for every } J \in \mathcal{D}, J \leq I\}\ .$$

To find $c_2$ one needs to compute $\sup_{x \in \Omega} B^d(x)$, where $\Omega = \{x = (x_1, x_2) : 1 \leq x_1 e^{-x_2} \leq c_1\}$. The domain is not convex, and apparently Theorem 3 is not applicable because of this, but we are going to find a supersolution in $\Omega$ which is concave in a wider - and convex - domain $\widetilde{\Omega} = \{x = (x_1, x_2) : 1 \leq x_1 e^{-x_2}\}$.

For such supersolutions $V$ inequality (11) still holds : $B^d \leq V$.

Here is a possible $V$ :

$$V(x_1, x_2) = 8c_1 \left(1 - \frac{e^{x_2}}{x_1}\right)\ .$$

In fact,

$$-d^2 V = 8c_1 \begin{pmatrix} \frac{2e^{x_2}}{x_1^3} & -\frac{e^{x_2}}{x_1^2} \\ -\frac{e^{x_2}}{x_1^2} & \frac{e^{x_2}}{x_1} \end{pmatrix} \geq 8c_1 \begin{pmatrix} \frac{e^{x_2}}{x_1^3}, & 0 \\ 0 & ,0 \end{pmatrix} \geq \frac{8c_1}{c_1} \begin{pmatrix} x_1^{-2} & 0 \\ 0 & 0 \end{pmatrix},$$

which means that $\langle -d^2 V\alpha,\alpha\rangle \geq \frac{8\alpha_1^2}{x_1^2}$ in $\Omega$. Concavity in $\widetilde{\Omega}$ is also clear.

Therefore,

$$c_2 \leq \sup_{\Omega} B^d \leq \sup_{\Omega} V \leq 8c_1(1-\frac{1}{c_1}) \ .$$

## 2.2 A two-weight inequality

$$\forall J\in\mathcal{D}\ \langle u\rangle_J\langle V\rangle_J \leq 1 \Rightarrow \forall I\in\mathcal{D}$$

$$\frac{1}{|I|}\sum_{\substack{\ell\leq I,\\ \ell\in\mathcal{D}}} |\langle u\rangle_{\ell+}-\langle u\rangle_{\ell}||\langle V\rangle_{\ell+}-\langle v\rangle_{\ell-}|\ ||\ell|$$

$$\leq C\langle u\rangle_I^{1/2}\langle v\rangle_I^{1/2}.$$

**Who moves ?**

$$x_1, x_2 = \langle u\rangle_J\ ,\ \langle V\rangle_J.$$

As in the previous problem $f^\alpha(x)$ is easy to find :

$$f^\alpha(x) = 4|\alpha_1|\ |\alpha_2|\ .$$

Bonus function $F\equiv 0$ here again.

Bellman equation

$$\sup_{\alpha=(\alpha_1,\alpha_2)}\left[\langle d^2 v\alpha,\alpha\rangle + 8|\alpha_1|\ |\alpha_2|\right] = 0, v\geq 0 \text{ in } \Omega = \{x=(x_1,x_2): 0\leq x_1, x_2; x_1x_2\leq 1\}\ .$$

Supersolution positive in $\Omega$ and concave in $\widetilde{\Omega} = \{x=(x_1,x_2): x_1\geq 0, x_2 \geq 0\}$ :

$$V(x_1,x_2) = A\sqrt{x_1x_2} - Bx_1x_2\ ,$$

with suitable $A, B (A=32, B=8$ works.)

$$C = \sup_{x\in\Omega}\frac{B^d(x_1,x_2)}{\sqrt{x_1x_2}} \leq \sup_{x\in\Omega}\frac{V(x_1,x_2)}{\sqrt{x_1x_2}} \leq 32\ .$$

## 2.3 John-Nirenberg inequality : Bellman equation with a drift but with $f^\alpha \equiv 0$.

$$\forall J \in \mathcal{D} \langle |\varphi - \langle \varphi \rangle_J|^2 \rangle_J \le \delta \Rightarrow \forall I \in \mathcal{D}$$
$$\langle e^\varphi \rangle_I \le C_\delta e^{\langle \varphi \rangle_I} \ .$$

**Who moves ?**

$$x_1 = \langle \varphi \rangle_J \ , \ x_2 = \langle |\varphi - \langle \varphi \rangle_J|^2 \rangle_J =$$

$$= \frac{1}{|J|} \sum_{\substack{I \le J \\ I \in \mathcal{D}}} \left\{ \frac{\langle \varphi \rangle_{I_+} - \langle \varphi \rangle_{I-}}{2} \right\}^2 |I| \ .$$

Notice that $(t = n)$

$$x_2^t = \mathbb{E}(x_2^{t+1} | x^t) = x_2 - \frac{x_2^+ + x_2^-}{2} = \left( \frac{x_1^+ - x_1^-}{2} \right)^2 = (\alpha_1^t)^2 \ .$$

On the other hand

$$x^{t+1} = x^t + \int_t^{t+1} \sigma dw^s + \int_t^{t+1} bds \ .$$

Thus drift $b$ stands for $\mathbb{E}(x^{t+1}|x^t) - x^t$ (in the case of discrete time). Therefore, $b(\alpha, x) = \begin{pmatrix} 0 \\ -\alpha_1^2 \end{pmatrix}$ in our case.

Notice that $f^\alpha \equiv 0$ as there is no $\frac{1}{|I|} \sum\limits_{\ell \le I} ...$ in the functional. Bellman equation in this case has a form

$$\sup_{\alpha = (\alpha_1, \alpha_2)} \left[ \frac{1}{2} \langle d^2 v \alpha, \alpha \rangle - \frac{\partial v}{\partial x_2} \alpha_1^2 \right] = 0 \ .$$

In other words :

$$\begin{pmatrix} \frac{\partial^2 v}{\partial x_1^2} - 2 \frac{\partial v}{\partial x_2} & \frac{\partial^2 v}{\partial x_1 \partial x_2} \\ \\ \frac{\partial^2 v}{\partial x_1 \partial x_2} & \frac{\partial^2 v}{\partial x_2^2} \end{pmatrix} \le 0, \tag{13}$$

in $\Omega_\delta = \{x = (x_1, x_2), x_1 \in \mathbb{R}, 0 \le x_2 \le \delta\}$.

The obstacle condition is

$$v(x) \ge F(x) \equiv e^{x_1} \text{ in } \Omega_\delta \tag{14}$$

Denote $B^d_\delta$ the dyadic Bellman function of a corresponding problem.

One can compute $v_\delta$ as well as try to use Theorems 2 and 3. Let us see what will happen. (What follows is taken from the work in preparation of L. Slavin and V. Vasyunin.)

First of all

$$v_\delta = \inf_{v_\delta \text{ is a supersolution}} \{V_\delta : V_\delta \geq e^{x_1} \text{ in } \Omega_\delta\}.$$

Secondly, such an infimum must have the following property : the matrix in (13) must be degenerate, namely

$$\left(\frac{\partial^2 v}{\partial x_1^2} - 2\frac{\partial v}{\partial x_2}\right)\left(\frac{\partial^2 v}{\partial x_2^2}\right) - \left(\frac{\partial^2 v}{\partial x_1 \partial x_2}\right)^2 = 0 \ . \tag{15}$$

The solutions of (15) which are greater than $e^{x_1}$ in $\Omega_\delta$ can be all written down.

Consider,

$$\varphi_{\varepsilon,q}(x_1,x_2) = q\frac{(1-\sqrt{\varepsilon - x_2})}{1-\sqrt{\varepsilon}} e^{x_1+\sqrt{\varepsilon-x_2}-\sqrt{\varepsilon}}, \delta \leq \varepsilon < 1, q \geq 1 \ .$$

They satisfy (15) and (13) in $\Omega_\delta$ and $\varphi_\varepsilon(x_1,x_2) \geq e^{x_1}$ in $\Omega_\varepsilon \geq \Omega_\delta$.

One can now easily see that

$$v_\delta = \inf_{\substack{\delta \leq \varepsilon < 1, \\ q \geq 1}} \varphi_{\varepsilon,q} = \varphi_\delta := \varphi_{\delta,1} \ .$$

Unfortunately, Theorem 2 is applicable :

$$B^d_\delta \geq v_\delta = \frac{1-\sqrt{\delta - x_2}}{1-\sqrt{\delta}} e^{x_1+\sqrt{\delta-x_2}-\sqrt{\delta}} \ , \tag{16}$$

but Theorem 3 is not applicable.

However, quite unexpectedly $v_\delta = \varphi_\delta$ is equal to non-dyadic Bellman function. Let us introduce it by imitating our dyadic definition

$$B_\delta(x_1,x_2) = \sup\{\langle e^\varphi\rangle_I : \langle\varphi\rangle_I = x_1 \ ,$$

$$\langle(\varphi - \langle\varphi\rangle_I)^2\rangle_I = x_2 \ , \ \sup_{J\leq I}\langle(\varphi-\langle\varphi\rangle_J)^2\rangle_J \leq \delta\} \ .$$

We call the attention of the reader to two things :

1) intervals $I, J$ are not dyadic anymore,

2) $\sup_{J\leq I}\langle(\varphi-\langle\varphi\rangle_J)^2\rangle_J$ is the norm squared of $\varphi$ in the space $BMO_2(J)$, $\|\varphi\|^2_{BMO_2(J)}$.

The fact that

$$B_\delta = v_\delta = \varphi_\delta$$

means that the constant of John Nirenberg for the norm $BMO_2$ is equal to 1 (if one looks at $\varphi_\delta$, one sees that it exists only for $0 \le \delta < 1$). One can also calculate the best constant in John-Nirenberg inequality $\langle e^{\varphi}\rangle \le C_\delta e^{\langle\varphi\rangle}$. Obviously, $C_\delta = \frac{e^{\sqrt{\delta}}}{1-\sqrt{\delta}}$.

## 2.4 Burkholder-Bellman function

$$\forall I \in \mathcal{D}\ |\langle g\rangle_{I+} - \langle g\rangle_{I-}| \le |\langle f\rangle_{I+} - \langle f\rangle_{I-}|$$
$$\Rightarrow \forall I \in \mathcal{D} \text{ such that } |\langle g\rangle_I| \le |\langle f\rangle_I|$$

one has

$$\langle |g|^p\rangle_I \le (p-1)^p \langle |f|^p\rangle_I \ , \ p \ge 2 \ .$$

The constant $(p-1)^p$ is sharp. This is a famous theorem of Burkholder which he proved by constructing the corresponding Bellman function. He found it by solving a corresponding Bellman PDE - a complicated one.

We would like to show a simple "heuristic" method of solution.

**Who moves ?**

$$x_1 = \langle g\rangle_J \ , \ x_2 = \langle f\rangle_J \ , \ x_3 = \langle |f|^p\rangle_J \ .$$

Our rules say that $f^\alpha(x) = 0$, $\mathbb{E}F(x_1^t, x_2^t, x_3^t) \approx \mathbb{E}|g|^p$. Denoting by $\mathcal{F}_t$ the $\sigma$-algebra generated by dyadic subintervals of $I$ of length $2^{-n}|I|, t = 2^n$, we can write $\mathbb{E}|g|^p \approx \mathbb{E}|(\mathbb{E}x_1|\mathcal{F}_t)|^p = \mathbb{E}|x_1^t|^p$ which gives us the correct bonus function $F(x_1, x_2, x_3) = |x_1|^p$.

Notice that $A = \{\alpha = (\alpha_1, \alpha_2, \alpha_3) : |\alpha_1| \le |\alpha_2|\}$ now.

This is because $|\alpha_1| = \frac{1}{2}|\langle g\rangle_{J+} - \langle g\rangle_{J-}|, |\alpha_2| = \frac{1}{2}|\langle f\rangle_{J+} - \langle f\rangle_{J-}|$, and we are given that the first quantity is always majorized by the second one.

So we have the Bellman equation

$$\sup_{\substack{|\alpha_1|\le|\alpha_2|,\\ \alpha_3}} \langle d^2 v\alpha, \alpha\rangle = 0$$

in $\Omega = \{x : (x_1, x_2, x_3) : |x_2|^p \le x_3\}$ (convex), with obstacle condition

$$v(x_1, x_2, x_3) \ge |x_1|^p \ .$$

We want to know what is the smallest $\beta$ such that

$$|x_1|^p \le v(x_1, x_2, x_3) \text{ in } \Omega,$$

$V$ satisfies (17), and

$$v(x_1, x_2, x_3) \le \beta x_3 \text{ in } \Omega \cap \{|x_1| \le |x_2|\}?$$

Burkholder found it ([3]) :

$$v(x_1, x_2, x_3) = p^{2-p}(p-1)^{p-1}(|x_1| + |x_2|)^{p-1} \;.$$

$$.\; (|x_1| - (p-1)|x_2|) + (p-1)^p x_3 \;,$$

and $\beta = (p-1)^p$. We will see now an “easy” method to find this Bellman function and the constant.

First step is the reduction of number of variables :

$$\varphi(x, y) = \sup_{z:(x,y,z)\in\Omega} [v(x, y, z) - \beta z] \;.$$

Properties : 1) $\sup_{\substack{\alpha=(\alpha_1,\alpha_2)\\ |\alpha_1|\le|\alpha_2|}} \langle d^2\varphi\alpha, \alpha\rangle = 0$ .

Explanation : the supremum of concave functions is not concave, but given a concave function one can get another concave function by taking the supremum over one of the variables.

2) $$\varphi(x, y) = \sup_{z\ge|y|^p} [v(x, y, z) - \beta z] \ge |x|^p - \beta|y|^p \;,$$

3) $$\varphi(\lambda x, \lambda y) = \lambda^p \varphi(x, y) \;.$$

Consider the quadratic form $\langle d^2\varphi\alpha, \alpha\rangle$ in a fixed point $(x, y) \in \mathbb{R}^2$. It is negative in a cone $\{|\alpha_1| \le |\alpha_2|\}$. We already saw in 2.3 that the Bellman function tends to have a degenerate quadratic form at each point. Assume that $\langle d^2\varphi\alpha, \alpha\rangle$ degenerates in $\{|\alpha_1| \le |\alpha_2|\}$. Then, obviously, it degenerates only on the boundary of this come (otherwise it would not be negative in the whole cone.)

By the third property of $\varphi$ it is unlikely that in say $I$ quadrant, the degeneration happens in northeast direction. In fact, the degeneration in this direction seems more pertinent to linear function then to a function which is homogeneous of order $p$.

So we may try to consider $\varphi$ with degeneration in northwest direction in the $I$ quadrant, northeast direction in the $II$ one, northwest again in the $III$ one, and northeast again in the $IV$ one. Such functions have the form

$$\varphi(x, y) = g(|x| + |y|)(|x| - \rho|y|)$$

with some unknown constant $\rho$ and unknown function $g$. But $g$ is "almost" known because $\varphi$ is homogeneous of order $p$ (its third property). Therefore,

$$\varphi_{\gamma,\rho}(x,y)=\gamma(|x|+|y|)^{p-1}(|x|-\rho|y|)\ ,$$

with unknown constants $\gamma,\rho$. And $\gamma\geq 0$.

**Lemma 2.1**

$$d^2\varphi_{\gamma,\rho}\leq 0\ \forall(x,y)\in\mathbb{R}^2,\forall|\alpha_1|\leq|\alpha_2|\Leftrightarrow\rho\geq p-1\ .$$

It is a direct calculation.

So $\rho\geq p-1$.

We must now find the smallest $\beta:=\tau^p$ such on $\mathbb{R}^2$

$$\gamma(|x|+|y|)^{p-1}(|x|-\rho|y|)\geq|x|^p-\tau^p|y|^p$$

with some $\gamma>0$, $\rho+1\geq P$.

Put $|x|+|y|=1\ ,\ |y|=S\in[0,1]$.

Minimum $\tau^p$ such that there exists $\gamma>0$ such that

$$\gamma(1-(\rho+1)s)-(1-s)^p+\tau^p s^p\geq 0,s\in[0,1]\ .$$

Take $s=\frac{1}{\rho+1}\leq\frac{1}{p}$. Then $\tau\geq p-1$.

Because we want the smallest $\tau$ let us try $\tau=p-1$, which forces also $\rho=p-1$.

Now $f_\gamma(s):=\gamma(1-ps)-(1-s)^p+(p-1)^p s^p$.

Does there exists $\gamma>0$ such that

$$f_\gamma(s)\geq 0\ ,\ s\in[0,1] \tag{17}$$

Notice that $f_\gamma(\frac{1}{p})=0$. Then to have (19) one needs $f'_\gamma(\frac{1}{p})=0$. This gives

$$\gamma_0=p^{2-p}(p-1)^{p-1}\ . \tag{18}$$

To check that (19) holds with $\gamma_0$ from (20) it is enough to check that $f''_{\gamma_0}\geq 0$ on $[t,1]$ with $t\leq\frac{1}{p}$.

But $f''_{\gamma_0}(s)=0$ implies $(\frac{1-s}{s})^{p-2}=(p-1)^p$, and so $s<\frac{1}{p}$.

We finally get all constants involved in $\varphi(x,y)$. After that we see that

$$v(x,y,z)=p^{2-p}(p-1)^p(|x|+|y|)^{p-1}(|x|-(p-1)|y|)$$

$$+(p-1)^pZ=\varphi(x,y)+\tau^pZ$$

is the function we need.

# 3 Conclusions

The Bellman functions built for one problem can be used for another one. Examples are numerous by now. See, for example, [4], [5]. Also one can combine Bellman functions using superposition to obtain the Bellman function of more and more complicated problems, see, for example, [6]. A thorough exposition of Bellman function technique, see in [7]. New creatures from the Bellman's zoo can be found in [8] along with the stochastic optimal control explanations.

In [9] one sees one more example of how this technique works in domains apparently far from its origins. Paper [10] gives another such example.

# Références


[1] St. Buckley, Summation conditions on weights, Mich. Math. J. 40 (1993), 153-170.

[2] R. Fefferman, C. Kenig, J. Pipher, The theory of weights and the Dirichlet problem for elliptic equations, Annals of Math, 134 (1991), 65-124.

[3] D.L. Burkholder, Boundary value problems and sharp inequalities for martingale transforms, Annals of Prob. 12 (1984), 647-702.

[4] F. Nazarov, A. Volberg, Heating of the Ahlfors-Beurling operator and estimates of its norms, Preprint.

[5] St. Petermichl, A. Volberg, Heating of the Ahlfors-Beurling operator : weakly quasiregular maps on the plane are quasiregular, To appear in Duke Math J.

[6] F. Nazarov, S. Treil, A. Volberg, The Bellman functions and two-weight inequalities for Haar multipliers, J. of the Amer. Math. Soc., 12 (1999), N°4, 909-928.

[7] F. Nazarov, S. Treil, The hunt for the Bellman function : applications to estimates of singular integral operators and to other classical problems in harmonic analysis, St Petersburg Math. J., 8 (1997), N°5, 32-162.

[8] F. Nazarov, S. Treil, A. Volberg, Bellman function in stochastic control and harmonic analysis, in "systems, Approximation, singular Integral operators, and related topics", ed. A. Borichev, N. Nikolski, OPERATOR THEORY : Advances and applications, v.129, 2001, 393-424, Birkhäuser Verlag.

[9] F. Nazarov, A. Volberg, The Bellman function and the imbeddings of the model space $K_\theta$, to appear in J. d'Analyse Math.

[10] S. Petermichl, J. Wittwer, A sharp weighted estimates on the norm of Hilbert transform via invariant $A_2$ characteristic of the weight, To appear in Mich. Math. J.

Université Paris VI
UFR de Mathématiques
4, place Jussieu
F - 75252 - Paris cedex 05