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Fonction de Correlation pour des Mesures Complexes et Principe du Maximum

(texte en anglais)

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(d'apres J. Sjöstrand and W.M. Wang)

Abstract: We study a class of holomorphic complex measures, which are close in an appropriate sense to a complex Gaussian. We show that these measures can be reduced to a product measure of real Gaussians with the aid of a maximum principle in the complex domain. The formulation of this problem has its origin in the study of a certain class of random Schrödinger operators, for which we show that the expectation value of the Green's function decays exponentially.

1. Introduction.

We study a class of normalized complex holomorphic measures of the form $e^{-\psi_n(x)}d^{2n}x$ in $\mathbf{R}^{2n}$, where $\psi_n(x)$ is holomorphic in x and $\operatorname{Re}\psi_n \geq 0$ and grows sufficiently fast at infinity, so that the integral is well defined. It is not presumed that $e^{-\psi_n(x)}d^{2n}x$ is a product measure. Moreover we assume that $e^{-\psi_n(x)}$ is "close", in some sense, to a complex Gaussian in certain regions of the complex space. Assuming that f does not grow too fast at infinity, we are interested in estimates of integrals of the form

$$\int f(x)e^{-\psi_n(x)}d^{2n}x,$$

which are *uniform* in n. So that eventually we can take the limit $n \to \infty$. Assume (for argument's sake) $|f(x)|_\infty = \mathcal{O}(1)$, then if $\psi_n(x)$ were real, we would immediately have

$$\int f(x)e^{-\psi_n(x)}d^{2n}x = \mathcal{O}(1)$$

uniformly in n. However it is clear that in the case $\psi_n(x)$ complex the same argument will not give us a bound which is uniform in n. Since typically,

$$\int |e^{-\psi_n(x)}|d^{2n}x \to \infty$$

as $n \to \infty$, even though

$$\int e^{-\psi_n(x)}d^{2n}x = 1,$$

for all n.

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In the following, we show that under appropriate conditions (convexity, domain of analyticity etc.), this class of measures can be reduced, **uniformly** with respect to the dimension of the space, to a product of real Gaussians. Hence the usual estimates of integrals with respect to positive measures become applicable.

The initial inspiration for this paper comes from random Schrödinger operators, where the expectation values of certain spectral quantities can be naturally expressed as the correlation functions of some normalized complex measures in even dimensions. Other examples of complex measures arise, for example, from considerations of analyticity of certain quantities in statistical mechanics. However for concreteness, we only state our results in the random Schrödinger case, although it is our belief that the method presented here should prove to be of a general nature, with possible applications to other fields.

We now describe the discrete random Schrödinger operator on $\ell^2(\mathbf{Z}^d)$:

$$H = t\Delta + V, \qquad (0 < t \leq 1) \tag{1.1}$$

where t is a parameter, Δ is the discrete Laplacian with matrix elements

$$\begin{aligned} \Delta_{i,j} &= 1 \qquad |i-j|_1 = 1, \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{1.2}$$

where $i,\ j \in \mathbf{Z}^d$, $|\cdot|_1$ is the ℓ^1 norm; V is a multiplication operator, $(Vu)(j) = v_j u_j$, with $v_j \in \mathbf{R}$. We assume that the v_j are independent random variables with a common distribution density g. We use $\langle \quad \rangle$ to denote the expectation with respect to (w.r.t.) the product probability measure. Such operators occur naturally in the quantum mechanical study of disordered systems. (See e.g. [FS,Sp].).

For small t, the spectrum of H is known to be almost surely pure point with exponentially localized eigenfunctions. (See e.g. [AM,DK,FMSS].) This is commonly known as *Anderson localization* after the physicist P. Anderson, who first realized the importance of the phenomenon [A]. Another related quantity of interest, which provides a necessary condition for the existing mechanisms for proving localization, is the density of states (d.o.s.). Roughly speaking, d.o.s. measures the number of states per unit energy per unit volume. More precisely, d.o.s. is the positive (non-random) Borel measure ρ such that

$$\langle \text{tr} f(H) \rangle = \int f(E) d\rho(E)$$

for all $f \in C_0(\mathbf{R})$. It is known generally that if g is smooth, then for t small enough or E large enough, ρ is also smooth. (See e.g. [CFS,BCKP].) In the continuum, one can prove similar results [W2], and moreover obtain an asymptotic expansion for ρ [W1].

Let Λ be a finite subset in $\mathbf{Z}^d$. Let Δ_Λ be the corresponding discrete Laplacian defined as in (1.2) for i, j in Λ. Define

$$H_\Lambda = t\Delta_\Lambda + V, \tag{1.3}$$

on $\ell^2(\Lambda)$. For E real, (assume $E \in \sigma(H_\Lambda)$ a.s.), let

$$G_\Lambda(E+i\eta) = (H_\Lambda - E - i\eta)^{-1}, \tag{1.4}$$

be the so called Green's function. We denote by $G_\Lambda(i,j;E+i\eta)$ the matrix elements of $G_\Lambda(E+i\eta)$. Then we have the following representation

$$\rho(E) = \lim_{\Lambda \nearrow \mathbf{Z}^d} \lim_{\eta \searrow 0} \mathrm{Im}\langle G_\Lambda(0,0;E+i\eta)\rangle \qquad a.s.$$

In this paper we study $\langle G_\Lambda(\mu,\nu;E+i\eta)\rangle$ for t sufficiently small or E sufficiently large. Our aim is to obtain estimates which are uniform in η, Λ, so that we can pass to the limit:

$$\langle G(\mu,\nu;E+i0)\rangle := \lim_{\Lambda \nearrow \mathbf{Z}^d} \lim_{\eta \searrow 0} \mathrm{Im}\langle G_\Lambda(\mu,\nu;E+i\eta)\rangle.$$

The existence of the limiting function can be obtained directly [SW] and we will not enter into the details here. Although the present method can give that too.

Assuming g is sufficiently smooth, using the supersymmetric representation of the inverse of a matrix, which was first used in this context in [BCKP], we can express $\langle G_\Lambda(\mu,\nu;E+i0)\rangle$ as a correlation function of a normalized complex measure. (For more details, see [SW1].) Let

$$\widehat{g}(\tau) = \int e^{-iv\tau} g(v)dv$$

denote the Fourier transform of g. Assume for example that $\widehat{g}(\tau) = e^{-k(\tau)} \neq 0$ for $\tau \in \mathbf{R}^+$, then (after taking the limit $\eta \searrow 0$)

$$\langle G_\Lambda(\mu,\nu;E+i0)\rangle = i\int x_\mu \cdot x_\nu [\det(iM_\Lambda) e^{-i(\sum_{|j-k|_1=1} tx_j\cdot x_k - \sum_j Ex_j\cdot x_j - i\sum_j k(x_j\cdot x_j))}] \prod_{j\in\Lambda} \frac{d^2x_j}{\pi}, \tag{1.5}$$

where $x_j \in \mathbf{R}^2$, $x_j \cdot x_k$ is the usual scalar product in $\mathbf{R}^2$ and

$$M_\Lambda = t\Delta_\Lambda - E - i\,\mathrm{diag}\,(k'(x_j\cdot x_j)), \tag{1.6}$$

where $\mathrm{diag}\,(k'(x_j\cdot x_j))$ denotes the diagonal matrix whose jj:th entry is $k'(x_j\cdot x_j)$. We notice the appearance of the Fourier transform of the original probability measure in the above induced measure. We believe that this is the main accomplishment of the supersymmetric representation here. After an integration by parts, (see [SW1]) we have further:

$$\langle G_\Lambda(\mu,\nu;E+i0)\rangle = \int M_\Lambda^{-1}(\mu,\nu;E)[\det(iM_\Lambda)e^{-i(\sum tx_j\cdot x_k - \sum Ex_j\cdot x_j - i\sum k(x_j\cdot x_j))}] \prod_{j\in\Lambda} d^2x_j. \tag{1.7}$$

Note that if the measure in the square brackets in (1.5), (1.7) were positive, then we would have immediately obtained that

$$|\langle G_\Lambda(\mu,\nu;E)\rangle| \le |M_\Lambda^{-1}(\mu,\nu;E)|_\infty$$

where the sup-norm is w.r.t. x. Hence the main idea is to make a change of contours in $(\mathbf{C}^2)^\Lambda$, so that on the new contour the measure becomes real positive. In order to do that we assume that g is such that $\widehat{g}$ is holomorphic in a region of $\mathbf{C}$ which includes the convex cone bounded by $\mathbf{R}^+$ and $e^{i\theta(E)}\mathbf{R}^+$, where $\theta(E) = \arg(1+iE) \subset]-\frac{\pi}{2}, \frac{\pi}{2}[$. Moreover we need to assume that g is ϵ $(0 \le \epsilon << 1)$ "close" to

$$g_0 = \frac{1}{\pi(1+v^2)},$$

so that there exists an open neighborhood $\Omega(E) \subset \mathbf{C}$ of $e^{i\theta(E)}[0,\infty[$ which is conic at infinity and in which $\widehat{g}$ is ϵ-close to g_0. (See [SW1] for more details.) For the precise conditions on g, see (2.1)-(2.3). Note that assuming t, ϵ small, then the final contour where the phase becomes real should be "close" to $((e^{\frac{i\theta(E)}{2}}\mathbf{R})^2)^\Lambda$. (Recall that $x_j \in \mathbf{R^2}$.) Therefore before we embark on the real work, we first rotate the contour from $(\mathbf{R}^2)^\Lambda$ to $((e^{\frac{i\theta(E)}{2}}\mathbf{R})^2)^\Lambda$. Using the assumptions on g, the measure then takes the simple form in (1.5), (1.7). Define

$$\phi := i(\sum_{|j-k|_1=1} tx_j \cdot x_k - \sum_j Ex_j \cdot x_j - i\sum_j k(x_j \cdot x_j)).$$

The change of contours is accomplished in two steps. We first look for a vector field v_t (holomorphic both in x and t) in $(\mathbf{C}^2)^\Lambda$ such that

$$\partial_t(e^{-\phi}) + \nabla_x(e^{-\phi}) \cdot v_t = 0, \tag{1.8}$$

or equivalently

$$\partial_t\phi + \nabla_x\phi \cdot v_t = 0. \tag{1.9}$$

where

$$v \cdot \nabla\phi := \sum_j (v_{j,1}\partial_{x_{j,1}}\phi + v_{j,2}\partial_{x_{j,2}}\phi).$$

Using the flow of the vector field to change variables, we get rid of the "interaction" term $\sum tx_j \cdot x_k$. The main difficulty here (as opposed to the case ϕ real) is to find v_t such that the corresponding flow stays in the appropriate region in $(\mathbf{C}^2)^\Lambda$ for t small enough so that the resulting integral is well defined and that the measure has no zeros there. This is achieved by using a cutoff function and solving (1.9) in some appropriate weighted space.

Unfortunately, after this operation, the coupling between x_j and x_k $(j \neq k)$ still persists in the Jacobian of the above "change of variables". Writing the measure as $e^{-L}\prod d^2x_j$ (with L holomorphic as the measure has no zeros there), we look for a second vector field ν_t (holomorphic in x and t) such that

$$\partial_t(e^{-L}) + \nabla_x(e^{-L}) \cdot \nu_t + e^{-L}\mathrm{div}\,\nu_t = 0.$$

or equivalently

$$\partial_t L + \nabla_x L \cdot \nu_t - \mathrm{div}\,\nu_t = 0. \tag{1.10}$$

We use a maximum principle in tube domains in the complex space to solve (1.10) under the condition that $\mathrm{Re}\,\mathrm{Hess}\, L > c > 0$ and some additional conditions on ∇L, which ensures that the resulting flow stays in tube domains around the real axis. This is in fact why we need to find the first vector field v_t to ensure that the new phase L is such that ∇L has the required properties.

Under these two changes of contours, the final measure takes the simple form

$$e^{-\sum_j z_j \cdot z_j} \prod_j \frac{d^2 z_j}{\pi}.$$

We then obtain that for $t/(|E|+1)$ sufficiently small and E in the appropriate range (depending on g), $\langle G_\Lambda(\mu, \nu; E + i\eta)\rangle$ decays exponentially in $|\mu - \nu|$ for all Λ sufficiently large, by using weighted estimates on $M_\Lambda^{-1}(\mu, \nu; E)$. The precise estimate is formulated in Theorem 2.1 in sect. 2.

We should mention here that the region of analyticity in t is uniform in Λ. The construction above does not depend on the fact that we have a nearest neighbour Laplacian (1.2). It works in the same way if Δ is replaced by any other symmetric matrix with off-diagonal matrix elements decay sufficiently fast.

As we have seen earlier $\langle G\rangle$ can be expressed as a correlation function of a normalized complex measure. In fact (1.5) shows clearly the link between the present problem and problems in statistical mechanics. (1.7) is special to the present problem. Our main constructions however do not depend on these special equalities arising from the symmetries of the present problem.

Before the first in a series of the works of B. Helffer and J. Sjöstrand [HS], where the equation (1.10), to our knowledge, first appeared in the context of statistical mechanics, one of the main tools to study correlation functions was cluster expansion–an algebraic way of rearranging the perturbation (e.g. in t) series. (1.10) provides an alternative way of treating such problems. The advantage, in our opinion, is that there is no combinatorics involved. The mathematics involved is purely analytical and self-contained. Moreover the convexity condition on L that one meets is the natural one.

Another general, but more probabilistic, approach to statistical mechanics is by using semi-groups or heat equations. It seems interesting to us to understand what would be the analogue of the construction presented here.

Although, as mentioned earlier, the inspiration for the present paper comes from quite a different source–random Schrödinger operators, in the end, the work presented here should be seen as a logical extension of the works of B. Helffer and J. Sjöstrand [HS,S1,S2] in statistical mechanics. (The work presented below might also be useful for the study of Feynmann formula.) Indeed one can take the standard example of studying the correlation function for the measure

$$\frac{e^{-\sum_{j,k\in\Lambda,|j-k|_1=1} t x_j \cdot x_k} \prod_{j\in\Lambda} e^{-k(x_j^2)} dx_j}{\int e^{-\sum_{j,k\in\Lambda,|j-k|_1=1} t x_j \cdot x_k} \prod_{j\in\Lambda} e^{-k(x_j^2)} dx_j}, \qquad x_j \in \mathbf{R},$$

assuming that k is such that the measure is well defined. It seems clear to us that under appropriate conditions on k, which essentially amounts to assuming k analytic and $k \neq 0$ on $\mathbf{R}^+$, k does not grow faster than linearly at infinity and some convexity conditions on k, the analyticity of the correlation function in t for small t should be a direct consequence of the constructions here.

2. Statement of the main result.

We first specify the class of densities g that we shall allow. Note that if g is the Cauchy distribution, $g_0(v) = \frac{1}{\pi}\frac{1}{v^2+1}$, then $k(\tau) = |\tau|$ for real τ and we have corresponding holomorphic extensions from each half axis (and we shall only use the one from the positive half axis, which is given by $k(\tau) = \tau$). We assume that g is of the form:

$$g(v) = (1 + \mathcal{O}(\epsilon))g_0(v) + r_\epsilon(v), \tag{2.1}$$

where

$$g_0(v) = \frac{1}{\pi}\frac{1}{v^2+1}$$

and r_ϵ has the following properties:

(a) r_ϵ is smooth and real on $\mathbf{R}$ and satisfies

$$\left|\frac{\partial^k r_\epsilon}{\partial v^k}\right| \leq C_k \epsilon \text{ for all } k \in \mathbf{N}, \tag{2.2}$$

for some fixed constants $C_0, C_1, ..$.

(b) There is a compact ϵ-independent set $K \subset \mathbf{C}$, symmetric around $\mathbf{R}$ with $i \notin K$, such that r_ϵ has a holomorphic extension to $\mathbf{C} \setminus K$ (also denoted by r_ϵ) with

$$r_\epsilon(v) = \mathcal{O}(\epsilon)\frac{1}{1+|v|^2} \text{ in } \mathbf{C} \setminus K. \tag{2.3}$$

The $\mathcal{O}(\epsilon)$ in (2.1) is determined by the requirement that $\int g(v)dv = 1$. Assuming also that $\epsilon \geq 0$ is small enough, as we shall always do in the following, we notice that it follows that $g(v) \geq 0$ and hence is a probability measure.

For all $\lambda > 2d$, introduce the convex open bounded set

$$W(\lambda) := \{\eta \in \mathbf{R}^d;\ 2\sum_1^d \cosh \eta_j < \lambda\}. \tag{2.4}$$

Let

$$p_\lambda(x) := \sup_{\eta \in W(\lambda)} x \cdot \eta \tag{2.5}$$

be the support function of $W(\lambda)$ so that $p_\lambda(x)$ is convex, even, positively homogeneous of degree 1. Moreover $p_\lambda(x) \geq 0$ with equality precisely at 0. In other words $p_\lambda(x)$ is a norm.

Equip the extended line $\overline{\mathbf{R}} := \{-\infty\} \cup \mathbf{R} \cup \{+\infty\}$ with the natural topology (i.e. the one induced from the topology on $[-1,+1]$ under the map $f : \overline{\mathbf{R}} \to [-1,1]$, where $f(\pm\infty) = \pm 1$, $f(x) = x/\sqrt{1+x^2}$, $x \in \mathbf{R}$). We define a subset $\mathcal{E} \subset \overline{\mathbf{R}}$ in the following way:

When $E \in \mathbf{R}$, we say that $E \in \mathcal{E}$ if and only if (iff) the following holds: The line L_E through $-i$ which is orthogonal to the vector $E+i$ (the direction of the segment joining $-i$ to E) does not intersect $K_- := \{z \in K; \operatorname{Im} z \le 0\}$ and separates K_- from E, in the sense that if P_+ is the open half-plane containing E with boundary L_E, and P_- the opposite open half-plane, then $K_- \subset P_-$.

When $E \in \{\pm\infty\}$, we say that $E \in \mathcal{E}$ iff the above holds with $L_E = i\mathbf{R}$.

Note that a necessary condition for $\mathcal{E}$ to be non-empty is that $-i$ does not belong to the convex hull of K_-. It is also clear that $\mathcal{E}$ is open and connected.

Let $d_{|E|}(\mu,\nu)$ be the distance on Λ associated to the norm $p_{|E|}(\mu-\nu)$, so that

$$d_{|E|}(\mu,\nu) = p_{|E|}(\mu-\nu)$$

when Λ is a finite set and

$$d_{|E|}(\mu,\nu) = \inf_{\widetilde{\mu}\in\pi_\Lambda^{-1}(\mu),\, \widetilde{\nu}\in\pi_\Lambda^{-1}(\nu)} p_{|E|}(\widetilde{\mu}-\widetilde{\nu}),$$

in the case when Λ is a torus, with $\pi_\Lambda : \mathbf{Z}^d \to \Lambda$ denoting the natural projection.

We can now state the main theorem.

Theorem 2.1. *For every $\mathcal{E}' \subset\subset \mathcal{E}$, there are constants $t_0 > 0$, $\epsilon_0 > 0$, such that if $0 \le \epsilon \le \epsilon_0$, $t \in]0,1]$, $E \in \mathcal{E}'$, $\frac{t}{|E+i|} \le t_0$, then for Λ sufficiently large we have uniformly in t, ϵ, E:*

$$|\langle G(\mu,\nu;E+i0)\rangle| \le \frac{1}{t}\exp(-d_{|E+i|/t}(\mu,\nu) + \mathcal{O}(\frac{t+\epsilon}{|E+i|})\rho(\mu,\nu)),\ \mu,\nu\in\Lambda. \tag{2.6}$$

Here ρ denotes the standard Euclidean distance in Λ.

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